Name:

Introduction to Quantum Mechanics

Final Exam

2002, Fall Quarter

1. **Operators as observables** (10 pt.).
State whether each answer to each question is true or false.
Note that more than one answer may be correct..

A. Observables are represented by hermitian operators *because*

- Those have eigenstates that are orthogonal to each other. FALSE (this is not true as eigenfunctions of unitary operators are also orthogonal to each other)
- Those have real eigenvalues. TRUE
- Two observables are simultaneously observable if the hermitian operators commute with each other. FALSE (again the same is true for unitary operators)
- Those have discrete spectra. FALSE

B. The coefficients of expansion of a state vector in eigenvectors of an observable

- Have no physical meaning. FALSE
- Have an interpretation as probability amplitudes. TRUE
- Have a physical meaning only if the spectrum is discrete. FALSE
- Have magnitude not larger than one TRUE (I accepted FALSE, as well, as this statement is only true for a discrete eigenvalue.

C. The product of uncertainties of two observables is

- Always nonvanishing. FALSE
- Not well defined if one of the observables is time. TRUE (I also accepted FALSE, as one can define the uncertainty in an indirect fashion)
- Vanishes if one of the observables is the Hamiltonian and the other observable does not depend on time explicitly. FALSE (take the other observable e.g. as the coordinate or momentum operator)

D. When measuring a physical quantity

- The possible values of measurement are the eigenvalues of the operator corresponding to the physical quantity. TRUE
- The average of the measurement can be expressed with the eigenvalues of the operator representing the physical quantity and the probabilities of measuring eigenvalues TRUE

- The result of the measurement is always the same (sharp) in an eigenstate of the Hamiltonian. FALSE
- Measurements, rapidly repeated each after the other, and performed on the same system provide the same result. FALSE (The first measurement alters the wave function and thus the result of the second measurement will already be different.)

2. **Matrices** (15pt)

A. Consider the matrix

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

State whether the following statements are false or true:

(a) M is symmetric FALSE

(b) M is hermitian TRUE

(c) M is real FALSE

(d) M is unitary TRUE

B. Consider the matrix

$$M = \begin{pmatrix} 0 & 2 & -3 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) How many eigenvalues does a 3×3 matrix have? 3 (as every 3×3 matrix does)

(b) How many *different* eigenvalues does M have? 1 ($\lambda = 0$)

(c) Provide an eigenvector of M. (1,0,0)

(d) Can M be diagonalized? (reason briefly)
No, if there is only one eigenvector the unitary matrix diagonalizing M cannot be constructed.

3. **Two level systems** (15 pt.).
(This is a modified version of problem 3.58, from the book)
Imagine a system in which there are just *two* independent states:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The most general state is a normalized linear combination:

$$|\Psi\rangle = a\,|1\rangle + b\,|2\rangle,$$

with

$$|a\,|^2 + |b\,|^2 = 1.$$

Suppose the Hamiltonian is

$$\boldsymbol{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix},$$

where h and g are real constants.

(a) Find the eigenvalues and eigenvectors of this Hamiltonian.

(b) Suppose the system starts out at $t = 0$ in state $|1\rangle$. What is the state at time t?

Answer:

$$|\psi(t)\rangle = e^{-iht/\hbar} \begin{pmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{pmatrix}.$$

(c) How does the answer change if the initial state is $|2\rangle$?
SOLUTION:
a) The eigenvalues are given by the characteristic equation:

$$\begin{vmatrix} h - E & g \\ g & h - \lambda \end{vmatrix} = (h - E)^2 - g^2 = 0$$

This gives $E_{1,2} = h \pm g$. The corresponding eigenvectors are obtained from $(h - E_{1,2})\alpha + g\beta = 0 = (\mp\alpha + \beta)g = 0$. Thus, $\beta = \pm\alpha$ and the eigenvectors are

$$|e_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |e_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

b) The vector $|\Psi(0)\rangle$ expanded in the eigenstates of the hamiltonian is $|\psi(0)\rangle = |1\rangle = (|e_1\rangle + |e_2\rangle)/\sqrt{2}$. Then at a later time

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left(|e_1\rangle e^{-iE_1 t/\hbar} + |e_2\rangle e^{-E_2 t/\hbar} \right) = e^{-iht/\hbar} \left[|1\rangle \cos(gt/\hbar) - i|2\rangle \sin(gt/\hbar) \right],$$

which agrees with the form, given as *'answer.'* (c) If the initial state is $|2\rangle$ then $|\psi(0)\rangle = |2\rangle = (|e_1\rangle - |e_2\rangle)/\sqrt{2}$. Then at a later time

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left(|e_1\rangle e^{-iE_1 t/\hbar} - |e_2\rangle e^{-E_2 t/\hbar} \right) = e^{-iht/\hbar} \left[|2\rangle \cos(gt/\hbar) - i|1\rangle \sin(gt/\hbar) \right],$$