

Physics 842
Particle Physics, Winter Term 2008
Midterm I

Thursday Feb 14th 2008, 9:30 am.

Professor Meadows

[To be handed in by 5 pm Tuesday Feb 19th, 2008]

Name

Solution to Question 1

In the "toy ABC" theory, there is a single vertex at which A, B and C can interact, and the masses are such that:

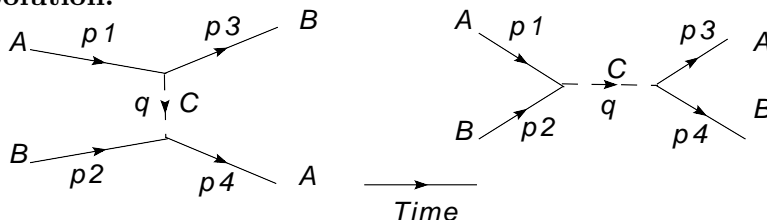
$$m_A > m_B > m_C ; m_A > (m_B + m_C)$$

- a) Is the scattering $A + B \rightarrow A + C$ possible in lowest order?

Solution: No. It is not possible to draw a Feynman diagram with these external particles consisting only of ABC vertices.

- b) Draw the lowest order diagrams for the reaction $A + B \rightarrow A + B$. [15 points]

Solution:



- c) Write down the matrix element for the reaction $A + B \rightarrow A + B$ for lowest order processes. [15 points]

Solution: For the first diagram:

$$\begin{aligned} -i\mathcal{M}_1 &= -i(2\pi)^4 g^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_C^2 c^2} \delta^4(p_1 - p_3 - q) \delta^4(p_2 + q - p_4) \\ &= -ig^2 \frac{1}{p_4 - p_2)^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \end{aligned}$$

Therefore,

$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_c^2 c^2}$$

For the second diagram:

$$\begin{aligned} -i\mathcal{M}_2 &= -i(2\pi)^4 g^2 \int \frac{d^4 q}{(2\pi^4)} \frac{1}{q^2 - m_c^2 c^2} \delta^4(p_1 + p_2 - q) \delta^4(q - p_3 - p_4) \\ &= -ig^2 \frac{1}{(p_1 + p_2)^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \end{aligned}$$

Therefore,

$$\mathcal{M}_2 = \frac{g^2}{(p_1 + p_2)^2 - m_c^2 c^2}$$

This gives:

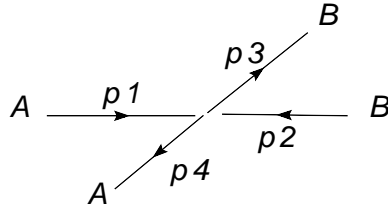
$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \frac{g^2}{(p_4 - p_2)^2 - m_c^2 c^2} + \frac{g^2}{(p_1 + p_2)^2 - m_c^2 c^2}$$

[15 points]

d) Compute $d\sigma/d\Omega$ for the reaction $A + B \rightarrow A + B$ in lowest order.

Solution: Make the computation in the CMS where

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|p_f|}{|p_i|}$$



In our case, $|p_f| = |p_i| = |p_{1-4}| = |p|$. Therefore,

$$\begin{aligned} (p_4 - p_2)^2 &= m_A^2 c^2 + m_B^2 c^2 - 2E_1 E_2 / c^2 + 2p^2 \cos \theta \\ (p_1 + p_2)^2 &= (E_1 + E_2)^2 / c^2 \equiv s \end{aligned}$$

Therefore

$$\frac{d\sigma}{d\Omega} = \frac{1}{s} \left(\frac{\hbar c}{8\pi} \right)^2 \frac{g^2}{m_A^2 c^2 + m_B^2 c^2 - 2E_1 E_2 / c^2 + 2p^2 \cos \theta - m_c^2 c^2} + \frac{g^2}{(E_1 + E_2)^2 / c^2 - m_c^2 c^2}$$

This simplifies if we assume $m_A = m_B = m$ and $m_C = 0$:

$$\frac{d\sigma}{d\Omega} = \left(\frac{g\hbar c}{2\pi s} \right)^2 [p^2(1 - \cos \theta) - s]$$

[15 points]

Solution to Question 2

- a) Write down the parity transformed Dirac spinor corresponding to

$$u(p) = \frac{Nc}{E + mc^2} \begin{pmatrix} E/c + mc \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix}$$

(Hint: What operator is used for P on Dirac spinors?)

Solution:

We follow the rule that a parity-transformed wave-function ψ' from the Dirac equation is related to its un-transformed state ψ by

$$\psi' = \gamma^0 \psi$$

From this we obtain, the present case:

$$u'(p) = \gamma^0 u(p) = \frac{Nc}{E + mc^2} \begin{pmatrix} E/c + mc \\ 0 \\ -p_z \\ -(p_x + ip_y) \end{pmatrix}$$

[15 points]

- b) Show that the spinor for an electron at rest is an eigenstate of parity.

Solution:

See below.

[15 points]

- c) Show that the spinor for a positron at rest is also an eigenstate of parity.

Solution:

A positron at rest has spinor has

$$\text{either } v^+(p) = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ (positive helicity) or}$$

$$v^-(p) = \frac{Nc}{E + mc^2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ (negative helicity).}$$

The parity operation on each of these yields:

$$\text{either } v^{+'} = \gamma^0 v^+(p) = N \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -v^+(p) \text{ (positive helicity) or}$$

$$v^{+'} = \gamma^0 v^-(p) = \frac{Nc}{E + mc^2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = -v^-(p) \text{ (negative helicity).}$$

d) How are the parity of the electron and positron related? [15 points]

Solution:

An electron at rest has spinor

$$\text{either } u^+(p) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (positive helicity) or}$$

$$u^-(p) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ (negative helicity).}$$

The parity operation on each of these yields:

$$\text{either } u^{+'} = \gamma^0 u^+(p) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = +u^+(p) \text{ (positive helicity) or}$$

$$u^{+'} = \gamma^0 u^-(p) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = +u^-(p) \text{ (negative helicity).}$$

From part (c) it is clear that, with this convention, the positron has parity $P_{e^+} = -1$ and the electron has $P_{e^-} = +1$. In any convention (we can choose $P = -\gamma^0$ rather than $P = \gamma^0$) the e^- and e^+ have opposite parity.

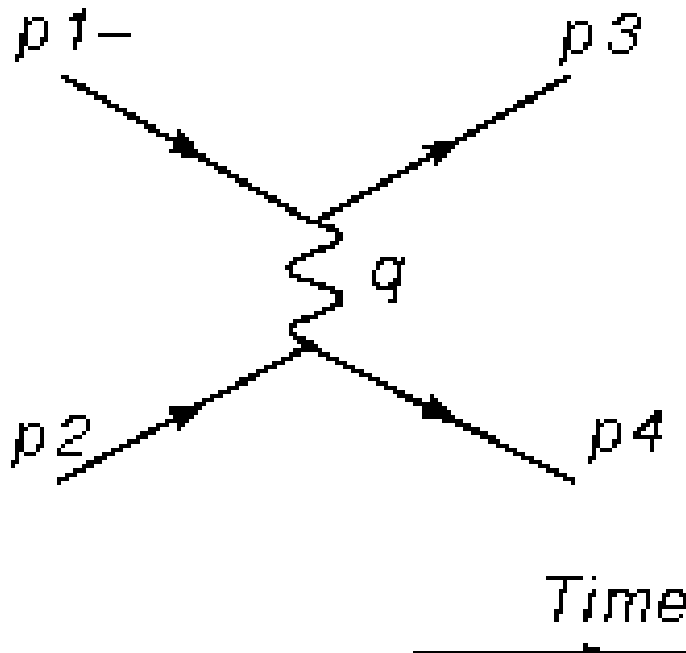
[15 points]

Solution to Question 3

- a) Sketch the lowest order diagram (more than one if you think there are more) contributing to elastic, electromagnetic scattering of a muon and a quark $\mu^- q \rightarrow \mu^- q$.

Are there any other diagrams ??

Solution:



- b) Use the Feynman rules to write down the matrix element \mathcal{M}_1 for the first diagram you sketched. [15 points]

Solution:

$$\begin{aligned}
 -i\mathcal{M}_1 &= (2\pi)^4 \int [\bar{u}(3)ig_e\gamma^\mu u(1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(4)ig_e\gamma^\nu u(2)] \\
 &\quad \times \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4)d^4q \\
 \mathcal{M}_1 &= -\frac{g_e^2}{q^2} [\bar{u}(3)\gamma^\mu u(1)] [\bar{u}(4)\gamma_\mu u(2)]
 \end{aligned}$$

where $q = (p_1 - p_3)$

[15 points]

- c) Use “Casimir’s trick” to evaluate the matrix element $\langle |\mathcal{M}_1|^2 \rangle$ summed over spins in the final state and averaged over the initial spins.

Solution:

“Casimir’s trick” sums over spins in the final state and also over initial state spins. In short, it converts a term like

$$\sum_{s_a, s_b} [\bar{u}^{s_a}(p_a)\Gamma_1 u^{s_b}(p_b)] [\bar{u}^{s_a}(p_a)\Gamma_2 u^{s_b}(p_b)]^*$$

where $\Gamma_{1,2}$ are 4 x 4 matrices to

$$Tr [\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]$$

where $\bar{\Gamma}_2 = \gamma^0 \Gamma_2^\dagger \gamma^0$.

Applying this to \mathcal{M}_1 where $\Gamma_2 = \gamma^\nu$ and $\bar{\Gamma}_2 = \gamma^0 \gamma^{\nu\dagger} \gamma^0 = \gamma^\nu$:

$$\begin{aligned} \mathcal{M}_1 &= -\frac{g_e^2}{(p_1 - p_3)^2} [\bar{u}(3)\gamma^\mu u(1)] [\bar{u}(4)\gamma_\mu u(2)] \\ \rightarrow \langle |\mathcal{M}_1|^2 \rangle &= \frac{g_e^4}{4(p_1 - p_3)^4} Tr [\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)] \\ &\quad \times Tr [\gamma_\mu(\not{p}_2 + mc)\gamma_\nu(\not{p}_4 + mc)] \end{aligned}$$

[For the other diagram, simply exchange $p_3 \leftrightarrow p_4$.] The factor 4 is equal to the number of initial spin states, $(2s_a + 1)(2s_b + 1)$, and is required to convert the sum to an average. [15 points]

- d) Evaluate this as a function of the 4-momenta p_{1-4} of the ingoing and outgoing particles.

Solution:

Expanding the first term:

$$Tr (\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + mc [Tr (\gamma^\mu \not{p}_1 \gamma^\nu) + Tr (\gamma^\mu \gamma^\nu \not{p}_3)] m^2 c^2 Tr (\gamma^\nu \gamma^\nu)$$

Using

$$Tr\{\text{prod. of odd number of } \gamma\text{'s}\} = 0,$$

the terms in mc are zero. Using

$$Tr\{\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\} = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})$$

the first term becomes

$$\begin{aligned} Tr(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3) &= p_{1\alpha}p_{3\beta}Tr(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta) \\ &= p_{1\alpha}p_{3\beta}4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu}) \\ &= 4(p_1^\mu p_3^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_3^\mu p_1^\nu) \end{aligned}$$

Finally, for the m^2c^2 term:

$$Tr(\gamma^\alpha\gamma^\beta) = 4g^{\mu\nu}$$

For the first factor in square parentheses, therefore, we obtain

$$Tr[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)] = 4[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2c^2 - p_1 \cdot p_3)]$$

The second factor in square parentheses is similar ($p_1 \rightarrow p_2$ and $p_3 \rightarrow p_4$) leading to

$$\begin{aligned} \langle |\mathcal{M}_1|^2 \rangle &= \frac{4g_e^4}{(p_1 - p_3)^4} [p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2c^2 - p_1 \cdot p_3)] \\ &\quad \times [p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}(m^2c^2 - p_2 \cdot p_4)] \\ &= \frac{4g_e^4}{(p_1 - p_3)^4} [2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) \\ &\quad - 2(m^2c^2)(p_2 \cdot p_4 + p_1 \cdot p_3) + 4(m^2c^2)^2] \end{aligned}$$

[For the other diagram, simply exchange $p_3 \leftrightarrow p_4$.] [15 points]

- e) Evaluate $\langle |\mathcal{M}_1|^2 \rangle$ for scattering in the CMS where the incoming momenta have magnitude $|\vec{p}| \gg mc$.

Solution:

In this frame

$$\begin{aligned} p_1 &= (E/c, 0, 0, p) ; p_2 = (E/c, 0, 0, -p) ; \\ p_3 &= (E/c, p \sin \theta, 0, p \cos \theta) ; p_4 = (E/c, -p \sin \theta, 0, -p \cos \theta) \end{aligned}$$

If, further $|\vec{p}| \gg mc$, then $E/c \approx p$ so that

$$(p_1 - p_3)^2 = -2p^2(1 - \cos \theta) = -4p^2 \sin^2 \frac{\theta}{2}$$

$$(p_1 - p_4)^2 = -2p^2(1 + \cos \theta) = -4p^2 \cos^2 \frac{\theta}{2} \quad \text{and}$$

$$p_1 \cdot p_2 = 2p^2 ; p_1 \cdot p_3 = p^2(1 - \cos \theta) ; p_1 \cdot p_4 = p^2(1 + \cos \theta) ;$$

$$p_3 \cdot p_4 = 2p^2 ; p_2 \cdot p_3 = p^2(1 + \cos \theta) ; p_2 \cdot p_4 = p^2(1 - \cos \theta)$$

Insert this into the expression for $\langle |\mathcal{M}_1|^2 \rangle$, neglecting the terms in $m^2 c^2$ in comparison with p^2 :

$$\begin{aligned} \langle |\mathcal{M}_1|^2 \rangle &= -\frac{g_e^4}{4 \sin^4(\theta/2)} [8 + 2(1 + \cos \theta)(1 + \cos \theta)] \\ &= -\frac{g_e^4}{2 \sin^4(\theta/2)} [5 + 2 \cos \theta + \cos^2 \theta] \end{aligned}$$

[15 points]