Physics 842
Particle Physics, Winter Term 2009
Midterm I

Thursday Feb 12th 2009, 9:30 am.

Professor Meadows

[To be handed in by 5 pm Monday Feb 16, 2009]

Name ....

Solution to Question 1

a) Write down the parity transformed Dirac spinor corresponding to

$$u(p) \quad = \quad \frac{Nc}{E+mc^2}\begin{pmatrix} E/c+mc \\ 0 \\ p_z \\ p_x+ip_y \end{pmatrix}$$

(Hint: What operator is used for P on Dirac spinors?)

Solution:

We follow the rule that a parity-transformed wave-function ψ' from the Dirac equation is related to its un-transformed state ψ by

$$\psi' = \gamma^0\psi$$

From this we obtain, the present case:

$$u'(p) = \gamma^0 u(p) \quad = \quad \frac{Nc}{E+mc^2}\begin{pmatrix} E/c+mc \\ 0 \\ -p_z \\ -(p_x+ip_y) \end{pmatrix}$$

NOTE that the factor

$$e^{-\frac{i}{\hbar}p_\mu x^\mu} \quad = \quad e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{r})}$$

does NOT change when $(x, y, z) \to (-x, -y, -z)$.

[15 points]

b) Show that the spinor for an electron at rest is an eigenstate of parity.

Solution:

See below.

[15 points]

c) Show that the spinor for a positron at rest is also an eigenstate of parity.

Solution:

A positron at rest has spinor has

$$\text{either} \quad v^{+}(p) \quad = \quad N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{(positive helicity) or}$$

$$v^{-}(p) \quad = \quad N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{(negative helicity).}$$

The parity operation on each of these yields:

$$\text{either} \quad v^{+\prime} = \gamma^0 v^{+}(p) \quad = \quad N \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -v^{+}(p) \quad \text{(positive helicity) or}$$

$$v^{+\prime} = \gamma^0 v^{-}(p) \quad = \quad N \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = -v^{-}(p) \quad \text{(negative helicity).}$$

[15 points]

d) How are the parity of the electron and positron related?

Solution:

An electron at rest has spinor

$$\text{either} \quad u^{+}(p) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{(positive helicity) or}$$

$$u^{-}(p) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{(negative helicity).}$$

The parity operation on each of these yields:

$$\text{either} \quad u^{+\prime} = \gamma^0 u^{+}(p) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = +u^{+}(p) \quad \text{(positive helicity) or}$$

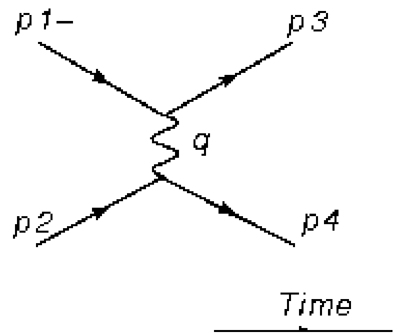
$$u^{+\prime} = \gamma^0 u^{-}(p) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = +u^{-}(p) \quad \text{(negative helicity).}$$

From part (c) it is clear that, with this convention, the positron has parity $P_{e^+} = -1$ and the electron has $P_{e^-} = +1$. In any convention (we can choose $P = -\gamma^0$ rather than $P = -\gamma^0$) the e^- and e^+ have opposite parity.

[15 points]

Solution to Question 2

Sketch the two lowest order diagrams contributing to $e^-e^- \to e^-e^-$ (Moller) scattering.
Solution:

Time $\rightarrow$  $+3 \leftrightarrow 4$

[15 points]

a) Use the Feynman rules to write down the matrix element $\mathcal{M}_1$ for the first diagram you sketched.

Solution:

$$
\begin{aligned}
-i\mathcal{M}_1 &= (2\pi)^4 \int [\bar{u}(3) i g_e \gamma^\mu u(1)] \frac{-i g_{\mu\nu}}{q^2} [\bar{u}(4) i g_e \gamma^\nu u(2)] \\
&\quad \times \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4) d^4 q \\
\mathcal{M}_1 &= -\frac{g_e^2}{q^2} [\bar{u}(3)\gamma^\mu u(1)] [\bar{u}(4)\gamma_\mu u(2)] \\
\text{where} &\quad q = (p_1 - p_3)
\end{aligned}
$$

[15 points]

b) Use the Feynman rules to write down the matrix element $\mathcal{M}_2$ for the second diagram.

Solution:

$$
\begin{aligned}
-i\mathcal{M}_2 &= (2\pi)^4 \int [\bar{u}(4) i g_e \gamma^\mu u(1)] \frac{-i g_{\mu\nu}}{q^2} [\bar{u}(3) i g_e \gamma^\nu u(2)] \\
&\quad \times \delta^4(p_1 - p_4 - q)\delta^4(p_2 + q - p_3) d^4 q
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_2 &= -\frac{g_e^2}{q^2}\,[\bar{u}(4)\gamma^\mu u(1)]\,[\bar{u}(3)\gamma_\mu u(2)] \\
\text{where} \qquad & q = (p_1 - p_4)
\end{aligned}
$$

[15 points]

c) Write $\mathcal{M}$, as the correct combination for these two matrix elements.

Solution:
Since the electrons in the final state are identical, we need to anti-symmetrize wrt their interchange. So we get

$$
\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2
$$

[15 points]

d) Expand $|\mathcal{M}|^2$, as a sum of terms in Dirac spinors.

Solution:

$$
\begin{aligned}
|\mathcal{M}|^2 &= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 - \mathcal{M}_1\mathcal{M}_2^* - \mathcal{M}_2\mathcal{M}_1^* \\
&= \frac{g_e^4}{(p_1-p_3)^4}|\,[\bar{u}(3)\gamma^\mu u(1)]\,[\bar{u}(4)\gamma_\mu u(2)]\,|^2 \\
&\quad + \frac{g_e^4}{(p_1-p_4)^4}|\,[\bar{u}(4)\gamma^\mu u(1)]\,[\bar{u}(3)\gamma_\mu u(2)]\,|^2 \\
&\quad - \frac{g_e^4}{(p_1-p_4)^2(p_1-p_3)^2} \\
&\quad \times \{[\bar{u}(3)\gamma^\mu u(1)]\,[\bar{u}(4)\gamma_\mu u(2)]\,[\bar{u}(4)\gamma^\mu u(1)]^*\,[\bar{u}(3)\gamma_\mu u(2)]^* \\
&\quad + [\bar{u}(4)\gamma^\mu u(1)]\,[\bar{u}(3)\gamma_\mu u(2)]\,[\bar{u}(3)\gamma^\mu u(1)]^*\,[\bar{u}(4)\gamma_\mu u(2)]^*\}
\end{aligned}
$$

[15 points]

Solution to Question 3

For $e^-e^- \to e^-e^-$ scattering introduced above:

a) Use "Casimir's trick" to evaluate the matrix element $< |\mathcal{M}_1|^2 >$ summed over spins in the final state and averaged over the initial spins.

Solution:
"Casimir's trick" sums over spins in the final state and also over initial state spins. In short, it converts a term like

$$\sum_{s_a,s_b} [\bar{u}^{s_a}(p_a)\Gamma_1 u^{s_b}(p_b)]\,[\bar{u}^{s_a}(p_a)\Gamma_2 u^{s_b}(p_b)]^*$$

where $\Gamma_{1,2}$ are 4 x 4 matrices to

$$Tr\,[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]$$

where $\bar{\Gamma}_2 = \gamma^0\Gamma_2^\dagger\gamma^0$.
Applying this to $\mathcal{M}_1$ where $\Gamma_2 = \gamma^\nu$ and $\bar{\Gamma}_2 = \gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$:

$$\begin{aligned}
\mathcal{M}_1 &= -\frac{g_e^2}{(p_1-p_3)^2}\,[\bar{u}(3)\gamma^\mu u(1)]\,[\bar{u}(4)\gamma_\mu u(2)] \\
\to\quad <|\mathcal{M}_1|^2> &= \frac{g_e^4}{4(p_1-p_3)^4} Tr\,[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_3+mc)] \\
&\quad \times\; Tr\,[\gamma_\mu(\not{p}_2+mc)\gamma_\nu(\not{p}_4+mc)]
\end{aligned}$$

[For the other diagram, simply exchange $p_3 \leftrightarrow p_4$.] The factor 4 is equal to the number of initial spin states, $(2s_a+1)(2s_b+1)$, and is required to convert the sum to an average. *[15 points]*

b) Evaluate this as a function of the 4-momenta p_{1-4} of the ingoing and outgoing electrons.

Solution:
Expanding the first term:

$$Tr\,(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3) + mc\,[Tr\,(\gamma^\mu\not{p}_1\gamma^\nu) + Tr\,(\gamma^\mu\gamma^\nu\not{p}_3)]\,m^2c^2 Tr\,(\gamma^\nu\gamma^\nu)$$

Using

$$Tr\{\text{prod. of odd number of }\gamma\text{'s}\} = 0,$$

the terms in mc are zero. Using

$$Tr\{\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\} = 4\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$$

the first term becomes

$$\begin{aligned} Tr\left(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3\right) &= p_{1\alpha}p_{3\beta}Tr\left(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right) \\ &= p_{1\alpha}p_{3\beta}4\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu}\right) \\ &= 4\left(p_1^\mu p_3^\nu - g^{\mu\nu}(p_1\cdot p_3) + p_3^\mu p_1^\nu\right) \end{aligned}$$

Finally, for the m^2c^2 term:

$$Tr(\gamma^\alpha\gamma^\beta) = 4g^{\mu\nu}$$

For the first factor in square parentheses, therefore, we obtain

$$Tr\left[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)\right] = 4\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2c^2 - p_1\cdot p_3)\right]$$

The second factor in square parentheses is similar ($p_1 \to p_2$ and $p_3 \to p_4$) leading to

$$\begin{aligned} <|\mathcal{M}_1|^2> &= \frac{4g_e^4}{(p_1-p_3)^4}\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2c^2 - p_1\cdot p_3)\right] \\ &\times \left[p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}(m^2c^2 - p_2\cdot p_4)\right] \\ &= \frac{4g_e^4}{(p_1-p_3)^4}\left[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)\right. \\ &- \left. 2(m^2c^2)(p_2\cdot p_4 + p_1\cdot p_3) + 4(m^2c^2)^2\right] \end{aligned}$$

[For the other diagram, simply exchange $p_3 \leftrightarrow p_4$.] *[15 points]*

c) Evaluate $<|\mathcal{M}_1|^2>$ for scattering in the CMS where the incoming momenta have magnitude $|\vec{p}| \gg mc$.

Solution:
In this frame

$$\begin{aligned} p_1 &= (E/c, 0, 0, p)\ ;\ p_2 = (E/c, 0, 0, -p)\ ; \\ p_3 &= (E/c, p\sin\theta, 0, p\cos\theta)\ ;\ p_4 = (E/c, -p\sin\theta, 0, -p\cos\theta) \end{aligned}$$

If, further $|\vec{p}| \gg mc$, then $E/c \approx p$ so that

$$\begin{aligned}
(p_1 - p_3)^2 &= -2p^2(1-\cos\theta) = -4p^2\sin^2\frac{\theta}{2} \\
(p_1 - p_4)^2 &= -2p^2(1+\cos\theta) = -4p^2\cos^2\frac{\theta}{2} \quad \text{and} \\
p_1 \cdot p_2 &= 2p^2 \ ; \ p_1 \cdot p_3 = p^2(1-\cos\theta) \ ; \ p_1 \cdot p_4 = p^2(1+\cos\theta) \ ; \\
p_3 \cdot p_4 &= 2p^2 \ ; \ p_2 \cdot p_3 = p^2(1+\cos\theta) \ ; \ p_2 \cdot p_4 = p^2(1-\cos\theta)
\end{aligned}$$

Insert this into the expression for $<|\mathcal{M}_1|^2>$, neglecting the terms in m^2c^2 in comparison with p^2:

$$\begin{aligned}
<|\mathcal{M}_1|^2> &= -\frac{g_e^4}{4\sin^4(\theta/2)}\left[8 + 2(1+\cos\theta)(1+\cos\theta)\right] \\
&= -\frac{g_e^4}{2\sin^4(\theta/2)}\left[5 + 2\cos\theta + \cos^2\theta\right]
\end{aligned}$$

For the other diagram, simply exchange $p_3 \leftrightarrow p_4$. This will give

$$\begin{aligned}
<|\mathcal{M}_2|^2> &= -\frac{g_e^4}{4\cos^4(\theta/2)}\left[8 + 2(1-\cos\theta)(1-\cos\theta)\right] \\
&= -\frac{g_e^4}{2\cos^4(\theta/2)}\left[5 - 2\cos\theta + \cos^2\theta\right]
\end{aligned}$$

[15 points]