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CHAOTIC QUANTUM BILLIARDS IN MAGNETIC FIELD: A SEMICLASSICAL ANALYSIS OF MESOSCOPIC EFFECTS

R.A. Serota
Department of Physics
University of Cincinnati
Cincinnati, Ohio 45221-0011

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Using the relationship between the correlation function of level density and classical probability of periodic motion, derived recently by Agraman, Imry, and Smilansky, we reinterpret the mesoscopic aspects of orbital response from the semiclassical perspective. We show that the semiclassical analysis conforms closely to the diagrammatic method and argue the equivalence of mesoscopic properties of small disordered metals and quantum chaotic billiards. Following Berry and Robnik, we consider Aharonov-Bohm billiards containing a magnetic flux line and generalize to the case of a uniform magnetic field.

Knowledge of spectral properties of quantum systems is the starting point for investigation of their thermodynamic properties. For instance, the dependence of level correlation function on magnetic flux in weakly disordered metals is used in perturbative derivations [1-3] of the fluctuations of orbital response and its canonical ensemble average. According to Efetov [4], the spectra of small metals conform to Wigner-Dyson statistics; he derived the level correlation function and showed that it coincides with that found from the random-matrix theory. [5] Later, the asymptotic behavior of the tail of the correlation function has been rederived by Altshuler and Shklovskii [6] using the perturbative diagrammatic technique. The latter is sufficient in the case of level broadening and/or temperature exceeding the mean interlevel spacing but the problems arise when the discreteness of the spectrum becomes relevant. Recently, we have derived [7] an expression for the orbital response for a range of experimentally significant temperatures and demonstrated a saturation to the zero-temperature dependence below the temperature of the order of mean interlevel spacing. Our analysis was based on the formula for the level correlation function interpolating between Efetov's result for discrete spectra and that of Ref. 6 for broad bands. The zero-temperature dependence of the persistent current on the flux has been independently obtained by Altland *et al* using Efetov's mapping to the supersymmetric non-linear σ-model. [8]

Interestingly, the statistical properties of energy spectra can be also obtained semiclassically with the help of the celebrated trace formula of Gutzwiller. [9] In the classically chaotic circumstance it yields the Wigner-Dyson statistics, while in the integrable circumstance it yields the Poisson statistics of the energy spectrum. [10] The semiclassical analysis is based on the Hannay-Ozorio de Almeida sum rule and the diagonal approximation which neglects quantum interference between different long periodic orbits but takes into account interference along the same orbit. We have argued that this approximation is equivalent to the perturbative derivation in the theory of disordered metals [7] and proposed that the latter can be used as the model system for the study of classically ergodic systems. For instance, the exact treatment of the non-linear σ-model in supersymmetric representation could shed light on the breakdown of the diagonal approximation and the relation of the even cumulants of the level density to the mean density imposed by the discreteness of the spectrum. [10]

Argaman, Imry, and Smilansky also argue that the diagonal approximation is equivalent to the perturbative treatment of disordered metals. [11] In addition, they derive a relationship between the classical probability of periodic motion and the spectral form factor. They proceed to apply this relationship to the semiclassical derivation of the orbital response of quasi-one-dimensional disordered rings and show that the results coincide with those obtained from the perturbation theory. In this work we first generalize their approach to general ergodic billiards containing a flux line and then to the case of a uniform magnetic field. Our analysis is based on the earlier work of Berry and Robnik. [12] We demonstrate a strong parallelism between the semiclassical derivation and perturbative diagrammatic approach. In the process we point to additional factors for the disordered metals to be viewed as a case study of classically ergodic behavior so that the techniques developed for its description, such as the supersymmetric non-linear σ-model, could be used advantageously in a more general meaning.

Following Argaman *et al*, the Fourier transform $K(E,t))$ of the level correlation function is related to the classical probability $p(E,t)$ to perform periodic motion at energy E via [11]

$$K(E,t) \cong \frac{2|t|}{(2\pi\hbar^2)}\frac{d\Omega}{dE}\, p(E,t), \tag{1}$$

where t is the period of the motion and Ω is the hyperspace volume. Assuming diffusive motion, the expression for the fluctuations of the persistent current in a ring of length L subject to the AB flux ϕ can be written as [11]

$$\langle I^2\rangle \cong c^2 \int_{-\infty}^{0} d\varepsilon\varepsilon \int_{-\infty}^{0} d\varepsilon'\varepsilon' \int_{-\infty}^{\infty} dt e^{it(\varepsilon-\varepsilon')/\hbar} \sum_{n=-\infty}^{\infty} \frac{|t|}{(2\pi\hbar)^2}\frac{L}{\sqrt{4\pi D|t|}}\exp\left(-\frac{n^2L^2}{4D|t|}\right)\left(\frac{2\pi n}{\phi_0}\right)^2\left(1-e^{i4\pi n\phi/\phi_0}\right), \tag{2}$$

where D is the diffusion coefficient and n is the winding number around the AB flux. Eq. (2) follows from [1] $I = -c\,\partial \int_{-\infty}^{0} d\varepsilon\varepsilon\nu(\varepsilon,\phi) \big/ \partial\phi$, the Gutzwiller's trace formula [9] for the density of states $\nu(\varepsilon,\phi)$, the account of the phase factor of $2\pi n\phi/\phi_0$ in the action of each classical trajectory, and Eq. (1) with the provision that the classical probability to perform periodic motion, corresponding to the winding number n, is

$$p_n(E_F,t) = \left(\frac{d\Omega}{dE}\right)^{-1}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-n^2/2\sigma^2\right), \quad \sigma^2 = 2D|t|/L^2 \tag{3}$$

We refer to Ref. 11 for further details of the calculation.

Two remarks should be made. First, the two terms in the last parentheses in Eq. (2) correspond to the diffuson and Cooperon contributions respectively in the perturbative diagrammatic approach. Second, the assumption of diffusive motion manifests itself only through the Gaussian distribution of winding numbers n. The latter is in complete analogy to the situation in ergodic chaotic billiards containing a flux line considered by Berry and Robnik. [12] They showed that the distribution of winding numbers is a discrete Gaussian,

$$\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{n^2}{2\sigma^2}\right), \quad \sigma^2 \approx \frac{\left(1-2^{-3/2}\right)\varsigma\left(\frac{3}{2}\right)}{2^{1/2}\pi^{3/2}}\frac{v|t|}{a^{1/2}} \equiv C\frac{v|t|}{a^{1/2}}, \tag{4}$$

where a is the billiard area and v is the velocity which, in the context of metals, should be identified with the Fermi velocity. Berry and Robnik used this result to find the average value of "the coefficient of time reversal symmetry breaking" – the second term in the last parentheses of Eq. (2). As a result, they derived the condition for the crossover to the unitary regime which coincides with that found previously by Efetov in the limit when the electron mean-free-path ℓ is such that $\ell \sim a^{1/2}$. In other words, the motion in the ergodic chaotic billiard can be viewed as diffusive with the understanding that $D \sim va^{1/2}$. This idea was first explored in the context of metallic particles with rough boundaries by Gorkov and Eliashberg. [13]

To establish a relationship with the perturbative diagrammatic calculations we use the Poisson formula to supplant the summation over the winding number with the summation over another variable (which we will here call "mode") in the spirit of Ref. 12. As a result, we find the following expression for the fluctuations of the total magnetic moment of the billiard:

$$\langle M^2\rangle = \left(\frac{\hbar a}{\phi_0}\right)^2 \int_{-\infty}^{\infty} dt \frac{1}{|t|^3}\sum_{m=-\infty}^{\infty}\sigma^2 \times \left[\begin{array}{l} e^{-8\pi^2\sigma^2\left(\frac{m}{2}\right)^2}\left(1-16\pi^2\sigma^2\left(\frac{m}{2}\right)^2\right) \\ -e^{-8\pi^2\sigma^2\left(\frac{m}{2}-\frac{\phi}{\phi_0}\right)^2}\left(1-16\pi^2\sigma^2\left(\frac{m}{2}-\frac{\phi}{\phi_0}\right)^2\right)\end{array}\right], \tag{5}$$

where σ^2 is defined in Eq. (4). The summation over all integer m's in Eq. (5) clearly underscores the fact that the AB response is periodic in magnetic flux with the period $\phi_0/2$. Therefore, it is sufficient to consider the interval $(-\phi_0/4, \phi_0/4]$. For the latter, only the term $m = 0$ need be retained; other terms are exponentially small in σ^2. It is widely believed, and is confirmed by direct calculations for disordered metals, [3,7] that the response to the uniform magnetic field in this flux interval is essentially identical to the AB response. Consequently, to generalize to the case of a uniform field we retain only the "zero mode [3,4,7]" in Eq. (5), $m = 0$, and set $\phi = Ha$. Furthermore, we consider the linear response regime, which is going to be defined shortly below, by expanding in ϕ.

Introducing the time scale $\tau \approx a^{1/2}/v_F$ and using the definitions of Eq. (4), we find the following expression for the fluctuation of the moment of the billiard:

$$\langle M^2\rangle \approx 8.8\mu_B^2(E_F\tau)^2\left(\frac{\phi}{\phi_0}\right)^2 \int_{t_{\min}}^{t_{\max}} \frac{dt}{t}. \tag{6}$$

In the process of derivation of Eq. (6) we assumed that $8\pi^2\sigma^2(\phi/\phi_0)^2 \lesssim 1$, that is $t \lesssim \tau_H \approx \left(1.7\pi^2\right)^{-1}\tau(\phi_0/\phi)^2$, which identifies τ_H as $t_{\max}$. The Gaussian distribution of winding numbers applies, obviously, to the periodic orbits traversing the billiard at least once. Therefore, τ should be identified with $t_{\min}$. Consequently, the integral in Eq. (6) takes on a form of $\ln(\tau_H/\tau)$, provided that $\tau \lesssim \tau_H$. The validity of linear response is,