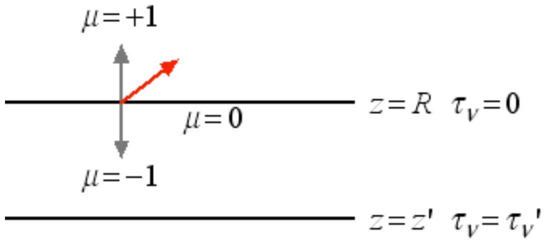
10 - GENERAL SOLUTION OF THE EQUATION OF TRANSFER

$$\frac{dI_\nu}{dz\sec\theta} = \cos\theta\frac{dI_\nu}{dz} = j_\nu\rho - k_\nu\rho I_\nu$$



It is conventional to set τ = 0 at z = R (the "surface" of the star), and have τ *increase inward* as z *decrease*s.

$$\text{i.e.,}\quad d\tau_\nu = -k_\nu\rho dz, \quad \tau_\nu = \int_R^{z'} -k_\nu\rho dz$$

We will also use $\mu = \cos\theta$ with $\mu = 1$ pointing outward from the stellar surface.

$$\mu\frac{dI_\nu}{dz} = j_\nu\rho - k_\nu\rho I_\nu \quad now \div by\ k_\nu\rho,$$

$$\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - \frac{j_\nu}{k_\nu} = I_\nu - S_\nu$$

$$where\ S_\nu \equiv \frac{j_\nu}{k_\nu} \quad the\ \text{"}Source\ Function\text{"}$$

$$\left[In\ TE, S_\nu = B_\nu \quad and \quad j_\nu = k_\nu B_\nu\right]$$

EQUATION OF (RADIATIVE) TRANSFER FOR STARS

$$\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \quad \text{"}The\ Equation\text{"}$$

Let's divide by μ, move I_ν over to the other side,

$$\frac{dI_\nu}{d\tau_\nu} - \frac{I_\nu}{\mu} = -\frac{S_\nu}{\mu}$$

For a general equation of the form $\frac{dy}{dx} + f(x)y = g(x)$ we can solve by multiplying through by $e^{\int f(x)dx}$ to get

$$\left(\frac{dI_\nu}{d\tau_\nu} - \frac{I_\nu}{\mu}\right)e^{-\tau_\nu/\mu} = -\frac{S_\nu}{\mu}e^{-\tau_\nu/\mu}$$

To solve this, we need some boundary conditions

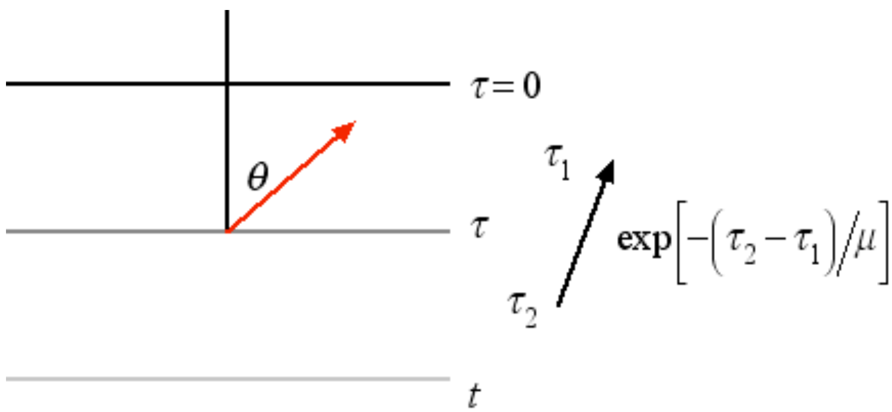
1. For outward flowing radiation:

$$\lim_{\tau \to \infty} I_\nu(\tau_\nu, \mu) e^{-\tau_\nu/\mu} \to 0$$

2. For inward flowing radiation:

$$I_\nu(0,\mu) = 0 \quad for \; -1 \le \mu \le 0$$

Let us drop the ν subscript for now (saves typing!)



Let t be the variable, and τ be the limits of the problem.

$$\frac{d}{dt}\left(I e^{-t/\mu} \right) = -\frac{S e^{-t/\mu}}{\mu}$$

$$I e^{-t/\mu} \Big|_{t=\tau_1}^{t=\tau_2} = -\int_{\tau_1}^{\tau_2} S(t) e^{-t/\mu} \frac{dt}{\mu} \qquad Then \times e^{-\tau_1/\mu} \; to \; get:$$

$$I(\tau_1,\mu) = I(\tau_2,\mu) e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S(t) e^{-(t-\tau_1)/\mu} \frac{dt}{\mu}$$

Example⇒ Emergent intensity for a semi-infinite atmosphere: Take $\tau_1 = 0, \tau_2 \to \infty$

$$I(0,\mu) = \int_0^\infty S(t) e^{-t/\mu} \frac{dt}{\mu}$$

The emergent intensity is a weighted mean over the source function, and corresponds to the integral of the fraction of the radiant energy at each layer, attenuated by extinction.

[NOTE for the mathematically-minded: *I is the Laplace Transform of S*]
Now, let us look at some simple, but illustrative examples.

If (for whatever reason) S is linear with optical depth, $S = a + bt$, then

$$I(0,\mu) = a + b\mu$$

For a finite slab of thickness τ, with no incident radiation, the emergent I is, for S = constant,

$$I(0,1) = S(1 - e^{-\tau})$$

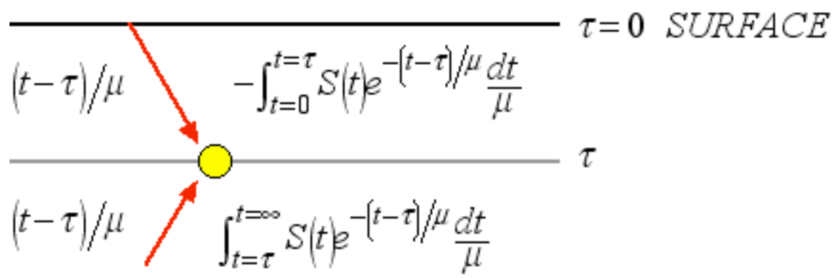
For $\tau >> 1$ (optically thick case),

$$I = S$$

For $\tau << 1$ (optically thin case)

$$I \approx S\tau \quad (where\ e^{-\tau} \approx 1 - \tau\ for\ \tau << 1)$$

Arbitrary Point $(\tau_1 = \tau)$ in a Semi-Infinite Atmosphere with No Incident Radiation.



FOR $\mu \geq 0$ *(OUTGOING RADIATION)* $(\tau_2 \to \infty)$

$$I(\tau,\mu) = \lim_{\tau_2 \to \infty} I(\tau_2,\mu)e^{-(\tau_2-\tau)/\mu} + \int_\tau^\infty S(t)e^{-(t-\tau)/\mu}\frac{dt}{\mu}$$

$$I(\tau,\mu) = \int_\tau^\infty S(t)e^{-(t-\tau)/\mu}\frac{dt}{\mu} \quad (0 \leq \mu \leq 1)$$

FOR $\mu \leq 0$ *(INCOMING RADIATION)* $(\tau_2 \to 0)$

$$I(\tau,\mu) = I(0,\mu)e^{\tau/\mu} + \int_\tau^0 S(t)e^{-(t-\tau)/\mu}\frac{dt}{\mu}$$

$$I(\tau,\mu) = -\int_0^\tau S(t)e^{-(\tau-t)/\mu}\frac{dt}{\mu} \quad (-1 \leq \mu \leq 0)$$

These are the "I Integrals"

J INTEGRAL

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu d\mu$$
$$= \frac{1}{2}\left[\int_0^1 \int_{\tau_\nu}^{\infty} S_\nu e^{-\left(t_\nu - \tau_\nu\right)/\mu} dt_\nu \frac{d\mu}{\mu} + \int_{-1}^{0} (-) \int_0^{\tau_\nu} S(t) e^{\left(\tau_\nu - t_\nu\right)/\mu} dt_\nu \frac{d\mu}{\mu}\right]$$

After some algebra that I won't bore you with,

$$J_\nu = \frac{1}{2}\left\{\int_{\tau_\nu}^{\infty} S_\nu E_1(t_\nu - \tau_\nu) dt_\nu + \int_0^{\tau_\nu} S_\nu E_1(\tau_\nu - t_\nu) dt_\nu\right\}$$

Where we have defined the Exponential integral of degree n as

$$E_n(x) = \int_1^{\infty} \frac{e^{-wx}}{w^n} dw$$

H INTEGRAL

$$H_\nu = \frac{1}{2}\left\{\int_{\tau_\nu}^{\infty} S_\nu E_2(t_\nu - \tau_\nu) dt_\nu - \int_0^{\tau} S_\nu E_2(\tau_\nu - t)_\nu dt\nu\right\}$$

K INTEGRAL

$$K_\nu = \frac{1}{2}\left\{\int_{\tau_\nu}^{\infty} S_\nu E_3(t_\nu - \tau_\nu) dt_\nu + \int_0^{\tau} S_\nu E_3(\tau_\nu - t)_\nu dt\nu\right\}$$

Useful relations for $E_n(x)$

$$E_n(x) = \frac{1}{n-1}\left[e^{-x} - xE_{n-1}(x)\right] \quad n \geq 2$$
$$E_1(x) = \frac{e^{-x}}{x}\left[1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} \cdots\right] \text{ for } x >> 1$$
$$E_1(x) = \gamma - \ln x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k k!} \text{ for } x > 0$$
$$(\gamma = 0.5572156.... \quad Euler's\ constant)$$