17 - PRELIMINARY MODELS

Let us estimate the values of some physical variables by considering the σ^{th} moment of the mass distribution weighted by $r^{-\nu}$.

$$I_{\sigma,\nu}(r_1) = \frac{G}{4\pi}\int_0^{r_1} \frac{\left[M(r)\right]^{\sigma}}{r^{\nu}} dM(r)$$

where $M(r)$ is the total mass interior to r: $M(r) = \int_0^r 4\pi r^2 \rho(r) dr$ from the mass continuity equation. Notice that we are now going to use $M(r)$ as independent variable, as I warned you earlier. This makes good physical sense because "stuff happens" where there is mass.

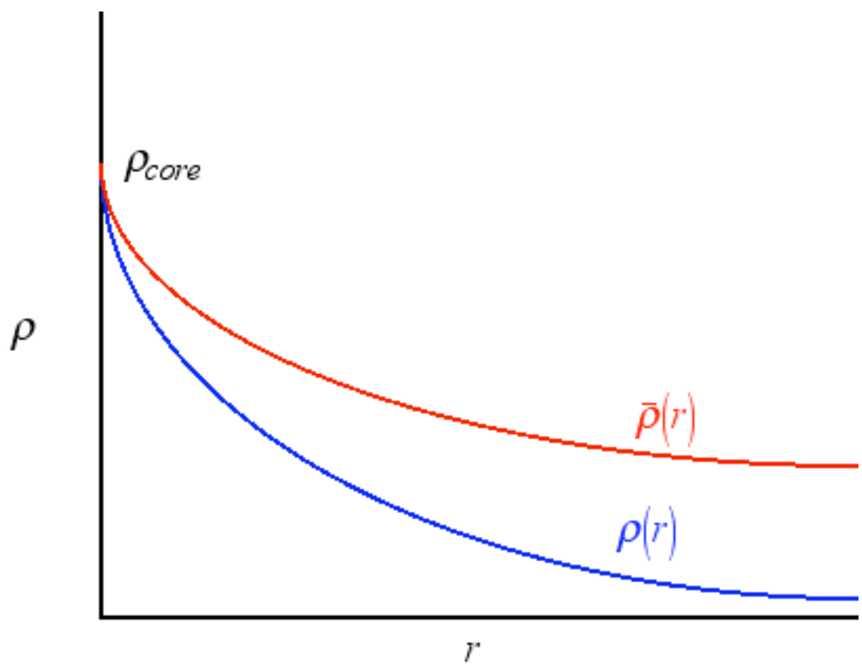
Let us also define the mean density interior to r as

$$\bar{\rho}(r) = \frac{3M(r)}{4\pi r^3}$$

If $\rho(r)$ decreases monotonically with r, then

$$\rho(r) \le \bar{\rho}(r)$$

It must also be true that $\bar{\rho}(r)$ decreases with r.



Since

$$\bar{\rho}(r) = \frac{M(r)}{\frac{4}{3}\pi r^3}, \quad \textit{and so} \quad r^\nu = \left[\frac{M(r)}{\frac{4}{3}\pi\bar{\rho}(r)}\right]^{\nu/3}$$

$$\textit{and} \quad I_{\sigma,\nu}(r_1) = \frac{G}{4\pi}\left(\frac{4\pi}{3}\right)^{\nu/3} \int_0^{r_1} \left[M(r)\right]^{\sigma-\nu/3} \left[\bar{\rho}(r)\right] dM(r)$$

If $\bar{\rho}(r)$ decreases from a maximum value of $\bar{\rho}(0) = \rho(0) = \rho_{core}$ to a minimum value of $\bar{\rho}(r_1)$ over the interval $(0, r_1)$ then

$$\underbrace{\frac{G}{4\pi}\left(\frac{4\pi}{3}\right)^{\nu/3} \rho_c^{\nu/3} \frac{M(r_1)^{\sigma+1-\nu/3}}{\left(\sigma+1-\nu/3\right)}}_{\substack{I_{\sigma,\nu} \textit{ using constant} \\ \bar{\rho}=\rho_c, \textit{ i.e., } I_{\sigma,\nu}(\max))}} \geq I_{\sigma,\nu}(r_1) \geq \underbrace{\frac{G}{4\pi}\left(\frac{4\pi}{3}\right)^{\nu/3} \bar{\rho}(r_1)^{\nu/3} \frac{M(r_1)^{\sigma+1-\nu/3}}{\left(\sigma+1-\nu/3\right)}}_{\substack{I_{\sigma,\nu} \textit{ using constant outer } \bar{r} \\ \textit{(minimum r possible, i.e., } I_{\sigma,\nu}\textit{(min))}}}$$

Using $I_{\sigma,\nu}$ *to get the mean values of P, T, and g weighted by mass (i.e., to get P, T, and g of the material, not just of the position) it is possible to show that:*

$$\bar{P} = \frac{\int_0^M P(r)dM(r)}{M} = \frac{I_{2,4}(R)}{M}$$

$$\bar{T} = \frac{\int_0^M T(r)dM(r)}{M} = \frac{4\pi\mu m_H}{3k}\frac{I_{1,1}(R)}{M}$$

$$\bar{g} = \frac{\int_0^M g(r)dM(r)}{M} = \frac{4\pi I_{1,2}(R)}{M}$$

$$\textit{Also, } P_c = I_{1,4}(R)$$

which yields

$$\bar{P} \geq \frac{3GM^2}{20\pi R^4} = 5.4x10^8 \left[\frac{M}{M_\odot}\right]^2 \left[\frac{R_\odot}{R}\right]^4 \textit{ atm.}$$

$$\bar{T} \geq \frac{G\mu m_H M}{5kR} = 4.61x10^6 \mu \left[\frac{M}{M_\odot}\right] \left[\frac{R_\odot}{R}\right] \; K$$

$$\bar{g} \geq \frac{3GM}{4R^2} = 2.05x10^4 \left[\frac{M}{M_\odot}\right] \left[\frac{R_\odot}{R}\right]^2 \textit{ cm s}^{-2}$$

$$\textit{Also } P_c \geq \frac{3G}{8\pi}\frac{M^2}{R^4} = 1.35x10^9 \left[\frac{M}{M_\odot}\right]^2 \left[\frac{R_\odot}{R}\right]^4 \textit{ atm.}$$

So, using just 3 equations, we have learned a lot about the mean pressure, temperature, and gravity in the interior of a star in terms of its mass M and radius R!

WORKED EXAMPLE FOR GRAD STUDENTS:

$$\bar{P} = \frac{\int_0^M P(r)dM(r)}{M} \text{ or } M\bar{P} = \int_0^M P(r)dM(r)$$

Integrate by parts and use the fact that $P(R) \approx 0$ *and* $M(0) = 0$

$$M\bar{P} = \left[\overbrace{P(r)}^{0\ at\ r=R} \underbrace{M(r)}_{0\ at\ r=0} \right]_0^R - \int_0^R M(r)\frac{dP(r)}{dM(r)}dM(r)$$

$$\text{But } \frac{dP}{dM} = -\frac{G}{4\pi}\frac{M(r)}{r^4} \text{ (combining HE and MC equations)}$$

$$\text{so } M\bar{P} = \frac{G}{4\pi}\int_0^R \frac{M^2(r)dM(r)}{r^4} = I_{2,4}(R)$$

$$M\bar{P} = I_{2,4}(R) \geq \frac{G}{4\pi}\left[\frac{4\pi}{3}\right]^{4/3}\left[\frac{M}{\frac{4}{3}\pi R^3}\right]^{4/3}\frac{M^{2+1+4/3}}{\left(2+1+4/3\right)}$$

$$M\bar{P} \geq \frac{3G}{20\pi}\frac{M^3}{R^4} \quad or$$

$$\bar{P} \geq \frac{3G}{20\pi}\frac{M^2}{R^4} = 5.4x10^8\left[\frac{M}{M_\odot}\right]^2\left[\frac{R_\odot}{R}\right]^4 \text{ atm.}$$

(*using* $R_\odot = 6.96x10^{10}$ *cm*, $M_\odot = 1.99x10^{33}$ *g*, 1 *atm.* $= 1.014x10^6$ *dyne cm*$^{-2}$)

Effects of Radiation Pressure

Before we go too much further, we should look at whether radiation pressure will or will not be the dominant form of pressure in cases where the gas is non-degenerate.

Define $\beta = \frac{P_g}{P_g + P_r} = \frac{P_g}{P_{total}}$. Then $P_g = \beta P_{total} = \frac{\rho kT}{\mu m_H}$ and $P_r = (1-\beta)P_{total} = \frac{1}{3}aT^4$.

If $T_g = T_r$ (which is true for TE), then

$$\frac{P_r}{P_g} = \frac{1-\beta}{\beta} = \frac{a}{3}\frac{\mu m_H}{k}\frac{T^3}{\rho} \text{ or } T = \left[\frac{1-\beta}{\beta}\frac{3}{a}\frac{k}{\mu m_H}\right]^{1/3}\rho^{1/3}.$$

Dropping the subscript on P_{total}:

$$P = \frac{1}{\beta}P_g = \frac{1}{\beta}\frac{\rho kT}{\mu m_H} = \left[\frac{1-\beta}{\beta^4}\left(\frac{k}{\mu m_H}\right)^4\frac{3}{a}\right]^{1/3}\rho^{4/3}$$

And at the core,

$$P_c = \left[\frac{1-\beta_c}{\beta_c{}^4}\left(\frac{k}{\mu m_H}\right)^4 \frac{3}{a}\right]^{1/3} \rho_c{}^{4/3}$$

But $P_c = I_{1,4}(R) \le \frac{G}{4\pi}\left(\frac{4\pi}{3}\right)^{4/3} \rho_c{}^{4/3} \frac{M^{2/3}}{\left(2/3\right)} = \frac{G}{2}\left(\frac{4\pi}{3}\right)^{4/3} \rho_c{}^{4/3} M^{2/3}$

so $\left[\frac{1-\beta_c}{\beta_c{}^4}\left(\frac{k}{\mu m_H}\right)^4 \frac{3}{a}\right]^{1/3} \rho_c{}^{4/3} \le \frac{G}{2}\left(\frac{4\pi}{3}\right)^{4/3} \rho_c{}^{4/3} M^{2/3}$

So $M \ge \left(\frac{6}{\pi}\right)^{1/2} \frac{1}{G^{2/3}}\left[\left(\frac{1-\beta_c}{\beta_c{}^4}\right)\left(\frac{k}{\mu m_H}\right)^4 \frac{3}{a}\right]^{1/2}$. *The biggest value this expression can have is* M *.*

Define β_* so that $M = \left(\frac{6}{\pi}\right)^{1/2} \frac{1}{G^{2/3}}\left[\left(\frac{1-\beta_*}{\beta_*{}^4}\right)\left(\frac{k}{\mu m_H}\right)^4 \frac{3}{a}\right]^{1/2}$, *maximizing the expression on the right-hand side*, so that $\frac{1-\beta_*}{\beta_*^4} \ge \frac{1-\beta_c}{\beta_c^4}$.

Since $\frac{1-\beta}{\beta^4}$ increases monotonically with $(1-\beta)$, $1-\beta_* \ge 1-\beta_c$

Thus, the mass of the star can be used to place an upper limit on $P_r/P_{total} = 1-\beta$ at the core of the star.

$\mu^2\left(\frac{M}{M_\odot}\right)$	$1-\beta_*$ (*upper limit on* $P_r/P_{total} = 1-\beta$)
0.56	0.01
0.81	0.02
2.14	0.10
3.83	0.20
9.62	0.30
122	0.80
∞	1.00

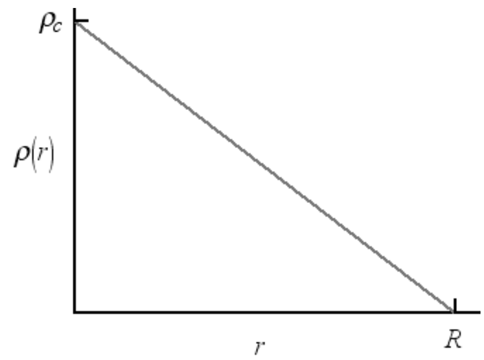
For the Sun, $1-\beta_c < 0.03 \quad (\mu_c \approx 1)$. (a rough mean for X<0.7 due to nucleosynthesis).

For α Aur (Capella) $(1-\beta_c) < 0.22 \quad (\mu_c \approx 1)$. $\quad (M = 4.2 M_\odot)$

Of course, we would eventually like to get a run of P, T, g, ρ, etc., as a function of r (actually M_r). We don't really have enough information to do this, but let's make a guess about ρ and see what we can do.

LINEAR MODEL

Suppose $\rho = \rho_c\left(1 - {}^{r}\!/\!_{R}\right)$,



Then the equations of hydrostatic equilibrium and mass continuity become

$$\frac{dP}{dr} = -\frac{M(r)G}{r^2}\rho_c\left(1 - {}^{r}\!/\!_{R}\right) \quad HE$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho_c\left(1 - {}^{r}\!/\!_{R}\right) \quad MC$$

so,

$$M(r) = \left(\frac{4\pi r^3}{3} - \frac{\pi r^4}{R}\right)\rho_c$$

$$Total\ \ M = M(R) = \left(\frac{4\pi R^3}{3} - \frac{\pi R^4}{R}\right)\rho_c = \frac{\pi R^3}{3}\rho_c$$

$$Eliminating\ \rho_c = \frac{3M}{\pi R^3}, \quad M(r) = M\left[\frac{4r^3}{R^3} - 3\frac{r^4}{R^4}\right]$$

$$so,\ \frac{dP}{dr} = -\frac{\pi G}{r^2}\rho_c^2\left(\frac{4}{3}r^3 - \frac{r^4}{R}\right)\left(1 - \frac{r}{R}\right) = -\pi G\rho_c^2\left[\frac{4}{3}r - \frac{7}{3}\frac{r^2}{R} + \frac{r^3}{R^2}\right]$$

$$or\ (upon\ integrating)\ P = P_c - \pi G\rho_c^2\left[\frac{2}{3}r^2 - \frac{7}{9}\frac{r^3}{R} + \frac{r^4}{4R^2}\right]$$

$$But\ P(R) = 0, \quad so\ \ P_c = \frac{5G}{4\pi}\frac{M^2}{R^4}$$

$$P(r) = \frac{5G}{4\pi}\frac{M^2}{R^4}\left[1 - \frac{24}{5}\left(\frac{r}{R}\right)^2 + \frac{28}{5}\left(\frac{r}{R}\right)^3 - \frac{9}{5}\left(\frac{r}{R}\right)^4\right]$$

$$Similarly,\ for\ P = P_g\ \left(i.e., \beta = 1\right), T = \frac{\mu m_H P}{\rho k}, so$$

$$T\left(r\right) = \frac{5\mu m_H G}{12k}\frac{M}{R}\left[1 + \frac{r}{R} - \frac{19}{5}\left(\frac{r}{R}\right)^2 + \frac{9}{5}\left(\frac{r}{R}\right)^3\right]$$