22 HOMOLOGY TRANSFORMATIONS

Before looking at the results of detailed models, it is useful to realize that once a detailed model exists, it is not always necessary to re-solve the entire model just to see what would happen if you were to vary a few parameters (mass, chemical composition, etc.), as long as the same basic equations apply.

Suppose you could transform one set of parameters into another by means of a multiplicative factor y^n:

$$
\begin{aligned}
r_2 &= y^{n_1} r_1 \\
M_{r_2} &= y^{n_2} M_{r_1} \\
\mu_2 &= y^{n_3} \mu_1 \\
P_2 &= y^{n_4} P_1 \\
\rho_2 &= y^{n_5} \rho_1 \\
T_2 &= y^{n_6} T_1 \\
L_{r_2} &= y^{n_7} L_{r_1} \\
k_{0,2} &= y^{n_8} k_{0,1} \\
\varepsilon_{0,2} &= y^{n_9} \varepsilon_{0,1}
\end{aligned}
$$

where the n's are constants. Such stars are said to be "homologous". This is often the case (at least to first order) if $P = \dfrac{\rho k T}{\beta \mu m_H}$ and $\beta = const.$

Under this condition, for example, the MC equation becomes

$$
\begin{aligned}
&dM_{r_2} = 4\pi r_2^2 \rho_2 dr_2 \\
&d\left(y^{n_2} M_{r_1}\right) = 4\pi \left(y^{n_1} r_1\right)^2 \left(y^{n_5} \rho_1\right) d\left(y^{n_1} r_1\right) \\
&y^{n_2}\left(dM_{r_1}\right) = y^{3n_1+n_5}\left(4\pi r_1^2 \rho_1 dr_1\right) \\
&y^{n_2} = y^{3n_1+n_5} \\
&n_2 = 3n_1 + n_5
\end{aligned}
$$

Similarly, it is possible to express the other equations of stellar structure in a similar manner.

MC $\overbrace{n_2}^{M_r} = \overbrace{3n_1}^{r^3} + \overbrace{n_5}^{\rho}$

HE $\overbrace{n_4}^{P} = \overbrace{n_2}^{M_r} + \overbrace{n_5}^{\rho} - \overbrace{n_1}^{r}$

Energy Generation $\overbrace{n_7}^{L} = \overbrace{3n_1}^{r^3} + \overbrace{2n_5}^{\rho^2} + \overbrace{n_9}^{\varepsilon_0} + \overbrace{\nu n_6}^{T^\nu}$

$\left(with\ \varepsilon = \varepsilon_0 \rho T^\nu\right)$

Radiative Energy Transport $\overbrace{n_6}^{T} = \overbrace{n_8}^{k_0} - \overbrace{n_1}^{r} + \overbrace{n_7}^{L} + \overbrace{\left(2-\alpha\right)n_5}^{\rho^{(1-\alpha)}} - \overbrace{\left(6+s\right)n_6}^{T^{-3-s}T^{-3}}$

$\left(with\ k = k_0 \rho^{1-\alpha} T^{-3-s}\right)$

Equation of State
$$\begin{cases} \left(\beta = 1\right) & \overbrace{n_4}^{P} = \overbrace{n_5}^{\rho} + \overbrace{n_6}^{T} - \overbrace{n_3}^{\mu} \\ \left(\beta = 0\right) & \overbrace{n_4}^{P} = \overbrace{4n_6}^{T^4}\ or\ \overbrace{3n_4 = 5n_5}^{P \propto \rho^{5/3}} \quad \left(redundant\ but\ often\ useful\right) \end{cases}$$

This gives 5 equations with 9 n's, so we need to specify 4 n's as independent variables.

The usefulness of these homology transformations comes when you want to estimate the relations between 2 (or more) variables.

Example

Consider just the Equation of State, HE and MC.

A star undergoing *uniform* contractions with little change in μ will have:

$$n_1 = 1 \quad \left(r_2 = y r_1\right)$$
$$n_2 = 0 \quad \left(M_2 = M_1\right)$$
$$n_3 = 0 \quad \left(\mu_2 = \mu_1\right)$$

Suppose the star contracts uniformly to a size that is one-half of its previous size (y=1/2).
Then we can solve for other n's:

$$n_5 = -3 \quad \Rightarrow \quad P_2 = 16 P_1$$
$$n_4 = -4 \quad \Rightarrow \quad \rho_2 = 8\rho_1$$
$$n_6 = -1 \quad \Rightarrow \quad T_2 = 2T_1$$

This type of transformation is called *Lane's Law*, and is sometimes used to describe certain phases of collapse during star formation.

For an ideal gas $(\beta = 1)$ with a Kramer's opacity $(\alpha = 0 \textit{ and } s = \frac{1}{2})$ producing energy using the pp chain $(\nu \sim 5)$ such as a star on the lower main sequence, one would get

$$L \propto M^{27/5} \sim M^{5.4}$$

For the upper main sequence, where radiation pressure dominates $(\beta = 0)$ and energy is produced by the CNO cycle $(\nu \sim 18)$,

$$L \propto M^{125/82} \sim M^{1.5}$$

The actual observed values are closer to 4.0 and 2.8, respectively. Obviously, this very crude method does not reproduce the stellar conditions in detail, but it does give some useful approximate trends.