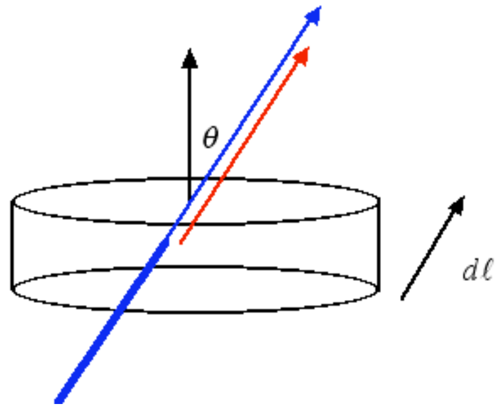
# 8 - EQUATION OF TRANSFER

Using conservation of energy and assuming a plane-parallel geometry (good for most situations):




Energy incident on disk + energy emitted within disk – energy absorbed by disk = energy exiting disk.

$$(I_\nu dA\cos\theta d\omega dt d\nu) + (j_\nu \rho dA\cos\theta d\ell d\omega dt d\nu) - (k_\nu I_\nu \rho dA\cos\theta d\ell d\omega dt d\nu)$$
$$= \left[(I_\nu + dI_\nu) dA\cos\theta d\omega dt d\nu\right]$$
$$so, (I_\nu) + (j_\nu \rho d\ell) - (k_\nu I_\nu \rho d\ell) = \left[(I_\nu + dI_\nu)\right]$$
$$and\ finally\ \frac{dI_\nu}{d\ell} = j_\nu \rho - k_\nu I_\nu \rho$$

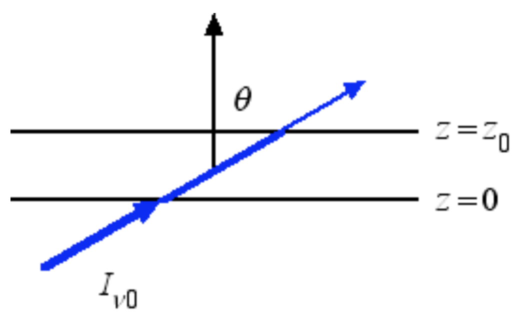
Examples:

1. Thermodynamic Equilibrium $I_\nu = B_\nu$ and $\frac{dI_\nu}{d\ell} = 0$

so, $j_\nu = k_\nu B_\nu$

The case of $\frac{dI_\nu}{d\ell} = 0$ is pretty boring, but sometimes we will assume that $j_\nu = k_\nu B_\nu$ is true *locally* (i.e. LTE).

2. Plane-parallel slab with $k_\nu = 0$, $j_\nu = constant$ (and assume $\rho = constant$) with energy incident equal to $I_{\nu,0}$ on one face.




$$start\ with\ \frac{dI_\nu}{d\ell} = j_\nu\rho - k_\nu I_\nu \rho$$

$$\ell = path\ length$$

$$\frac{dI}{d\ell} = j\rho \quad d\ell = dz\sec\theta$$

$$\frac{dI}{dz\sec\theta} = \cos\theta\frac{dI}{dz} = j\rho$$

$$\cos\theta\frac{dI_\nu}{dz} = j_\nu\rho$$

$$dI_\nu = j_\nu\rho\sec\theta dz$$

$$I_\nu = j_\nu\rho\sec\theta z + "const."$$

case a:

$$\theta\ between\ 0\ and\ \frac{\pi}{2}$$

$$at\ z = 0,\ I_\nu = I_{\nu,0} \quad so \quad "const." = I_{\nu,0}$$

$$I_\nu = I_{\nu,0} + j_\nu\rho z\sec\theta \quad for\ 0 < z < z_0$$

case b:

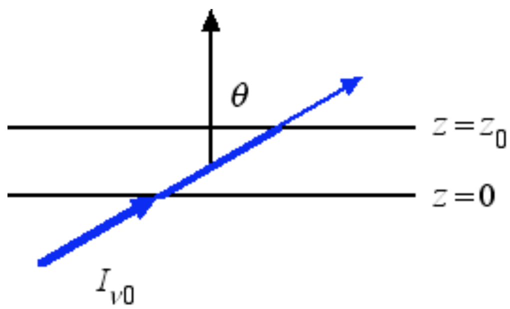
$$\theta\ between\ \frac{\pi}{2}\ and\ \pi\ (downward)$$

$$at\ z = z_0,\ I_\nu(downward) = 0$$

$$"const." = 0 - j_\nu\rho z_0\sec\theta$$

$$I_\nu = -j_\nu\rho\sec\theta(z_0 - z) \quad for\ 0 < z < z_0$$

$$(\sec\theta\ is\ negative)$$




3. $k_\nu = constant\ and\ j_\nu = 0 \quad (\rho = const.)$

$$\cos\theta \frac{dI_\nu}{dz} = -k_\nu \rho I_\nu$$

$$d\ln I = \frac{dI}{I} = -k\rho \sec\theta dz$$

$$\ln I_\nu = -k_\nu \rho (\sec\theta) z + "const."$$

$$or\ I_\nu = const \cdot e^{-\kappa_\nu \rho z \sec\theta}$$

case a: $0 < \theta < \frac{\pi}{2}$

$$at\ z = 0,\ I_\nu = I_{\nu,0}$$

$$I_\nu = I_{\nu,0} e^{-k_\nu \rho \sec\theta\, z}$$

case b: $\frac{\pi}{2} < \theta < \pi$

$$at\ z = z_0,\ I_\nu = 0$$

$$\ln(0) = -k_\nu \rho \sec\theta\, z_0 + "const."$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(-\infty) \qquad so \qquad \Rightarrow (-\infty)$$

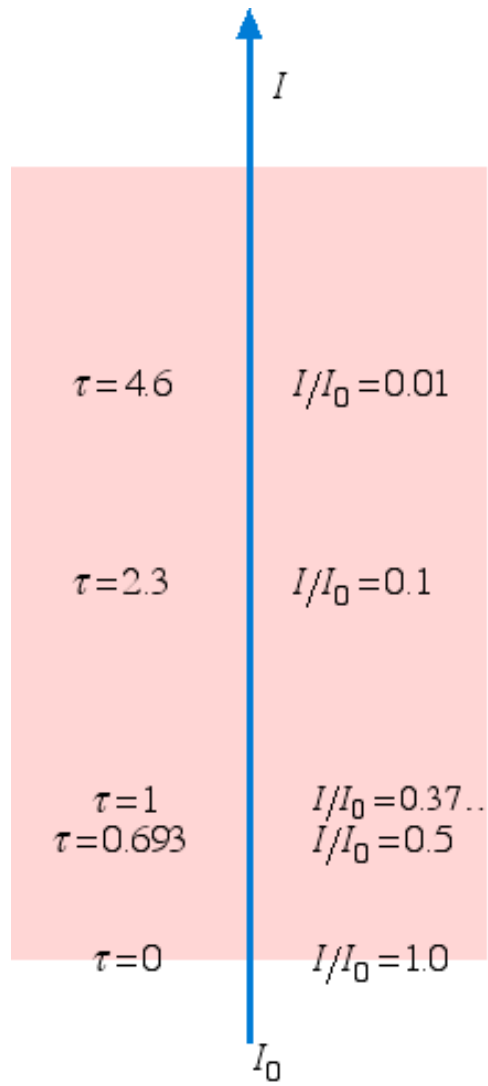
$$I_\nu = e^{-k_\nu \rho \sec\theta\, z - \infty} = 0$$

i.e., if there is no incident radiation in that direction, and none is emitted in the slab, you don't see anything!

Define $\tau_\nu \equiv \int_0^z k_\nu \rho dz$ *and so* $d\tau_\nu = k_\nu \rho dz$. This $\tau_\nu$ is what we will call the "optical depth" (here defined along the normal, not the actual path length) within the slab. For the previous case 3a (pure extinction) $\frac{I_\nu}{I_{\nu,0}} = e^{-\tau_\nu \sec\theta}$.

For $\sec\theta = 1$ (vertical)

| $\tau_\nu$ | $I_\nu / I_{\nu,0}$ |
|---|---|
| 0 | 1.0 |
| 0.693 | 0.50 |
| 1 | 0.37 |
| 2.3 | 0.10 |
| 4.6 | 0.01 |




TWO NOTES:

1. We will later define $\tau$ versus z for stars such that $\tau = 0$ at the surface, increasing inward, with dz *positive* going outwards (i.e. $d\tau \propto -dz$).
2. We will refer to

   $\tau << 1$ as "optically thin"
   $\tau >> 1$ as "optically thick"