

Review of Special Relativity and Relativistic Kinematics

- **Space and time form a Lorentz four-vector (t, \vec{x}) .**
- **The spacetime point (t, \vec{x}) which describes an event in one inertial reference frame and the spacetime point (t', \vec{x}') which describes the same event in another inertial reference frame are related by a Lorentz transformation.**
- **Energy and momentum form a Lorentz four-vector (E, \vec{p}) we call the four-momentum.**
- **The four-momentum of an object in one inertial frame is related to the four-momentum in another inertial frame by a Lorentz transformation.**
- **Energy and momentum are conserved in all inertial frames.**

Some Notation

- The components of a four vector will be denoted by

$$a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3)$$

- $c = 1$; $\vec{z} \parallel \hat{z}$

- $\gamma = \sqrt{\frac{1}{1 - \beta^2}}$

- $$\left. \begin{aligned} (a^0)' &= \gamma (a^0 - \beta a^3) \\ (a^1)' &= a^1 \\ (a^2)' &= a^2 \\ (a^3)' &= \gamma (a^3 - \beta a^0) \end{aligned} \right\} \text{Lorentz Transformation}$$

Lorentz Invariants

- We define the covariant vector a_μ in terms of the components of its cousin, the contravariant vector a^μ

$$a_\mu = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3)$$

- The dot product of two four vectors a and b is defined to be:

$$a \cdot b = a_\mu b^\mu = a_\mu b^\mu = (a^0)^2 - (\vec{a})^2$$

- By explicit calculation, we can find that $a \cdot b$ is **Lorentz invariant**, i.e. ,

$$a' \cdot b' = a \cdot b$$

An Example, Muon Decay

- We are going to watch a muon (μ) decay. In its own rest frame, this will take 10^{-6} seconds. In the laboratory, let the muon move with $v = 0.9999$ which gives $\gamma = 70.71$.



- In the rest frame of the muon, the laboratory is moving with $v = -0.9999$.
- In its own rest frame, it is born at $t = t_0; z = 0$ and it dies at $t = t_1; z = 0$ with $t_1 - t_0 = 10^{-6}$ seconds.
- In the laboratory we calculate its birth and death times

$$t_0 = \gamma(t_0' - v z_0'); \quad t_1 = \gamma(t_1' - v z_1')$$

- From which we calculate the lifetime in the laboratory

$$t_{decay} = t_1 - t_0 = \gamma(t_1' - t_0') = 70.71 \times 10^{-6} \text{ seconds}$$

Muon Decay, continued

- We usually refer the decay time in the particle's rest frame as its **proper time** which we denote τ . In its rest frame $\vec{x}_0 = \vec{x}_1 = 0$, so $\Delta s^2 = x_\mu x^\mu$ which should be Lorentz invariant. Let's check this explicitly.

- In the laboratory, the muon is traveling with speed $v = 0.9999$ and it travels for $t_{decay} = 70.71 \times 10^{-6}$ seconds, so the distance traveled will be

$$c t_{decay} = 0.9999 \times 70.71 \text{ seconds}$$

- This gives $\tau = \sqrt{1 - 0.9999^2} \times 70.71 \times 10^{-6}$ seconds, which gives $\tau = 1 \times 10^{-6}$ seconds.

Even More Muon Decay Physics

- We just found the decay distance in the laboratory to be

- $c t_{decay} = 0.9999\ 70.71$ seconds

- If you are not comfortable with measuring distance in seconds, use $c = 3 \times 10^8$ cm/sec rather than $c = 1$ to get

$$c t_{decay} = 2.4 \times 10^6 \text{ cm}$$

- In English units, 1nanosecond \sim 1 foot (30cm).

Four- momentum

- We denote the four- vector corresponding to energy and momentum

$$p^\mu = (E, p_x, p_y, p_z) = (E, \vec{p})$$

- Because we expect this to be a Lorentz four- vector, $p_\mu p^\mu$ should be Lorentz invariant. We do the calculation for the general case, and then specialize to the center- of- momentum frame (where the object is at rest, so has momentum zero).

$$p_\mu p^\mu = E^2 - \vec{p}^2 = E_{rest}^2 = m^2$$

- If we do not set $c = 1$,

$$p_\mu p^\mu = E^2 - \vec{p}^2 c^2 = E_{rest}^2 = m^2 c^4$$

Four-momentum, continued

- The equation $E^2 = p^2 + m^2$ was “derived” assuming that a particle at rest has zero momentum. But what about a particle with no mass? Classically, the less mass a particle has, the lower its momentum: $p = mv$, so a massless particle would have zero momentum. Relativistically, this is no longer true. A massless particle can have any energy as long as $v = c$, in which case we can satisfy $E = |\vec{p}|c$ for any value of E .
- For a particle at rest with mass m we can find the energy and momentum in any other inertial frame using a Lorentz transformation (note: if the particle is moving with velocity v in the \hat{z} direction, the laboratory is moving in the $-\hat{z}$ direction according to the particle):

$$E = p^0 = (\gamma p^0 + \gamma v p^3) = \gamma m$$

$$|\vec{p}| = p^3 = (\gamma p^3 + \gamma v p^0) = \gamma m v$$

Classical Limits: Energy

- Let's do a Taylor series expansion for $E(x)$. This should have the form

$$f(x) = f(0) + \frac{df}{dx} x + \frac{1}{2} \frac{d^2 f}{dx^2} x^2 +$$

- With $f(x) = E(x)$, we calculate $\frac{dE}{dx}$ and $\frac{d^2 E}{dx^2}$ and take the limits $x \rightarrow 0$:

$$\frac{dE}{dx} = m \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \quad \text{at } x=0$$

$$\frac{d^2 E}{dx^2} = m \left(1 + \frac{2v^2}{c^2}\right) \left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}} \quad \text{at } x=0$$

$$E(x) = m + 0 + \frac{1}{2} m \frac{v^2}{c^2} +$$

Classical Limits: Energy, continued

- With $f(x) = E(x^2)$, we calculate $\frac{dE}{d(x^2)}$ and $\frac{d^2E}{d(x^2)^2}$ and take the limits $\rightarrow 0$:

$$\frac{dE}{d(x^2)} = \frac{1}{2}m(1 - x^2)^{-\frac{3}{2}} \rightarrow \frac{1}{2}m$$

$$\frac{d^2E}{d(x^2)^2} = \frac{3}{4}m(1 - x^2)^{-\frac{5}{2}} \rightarrow \frac{3}{4}m$$

$$\begin{aligned} E(x^2) &= m + \frac{1}{2}m x^2 + \frac{1}{2} \frac{3}{4}m x^4 + \dots \\ &= m + \frac{1}{2}m x^2 + \frac{3}{8}m x^4 + \dots \end{aligned}$$

Classical Limits: Momentum

- Let's start by writing momentum in terms of β :

$$p = m \gamma v = m \left(1 - \beta^2\right)^{-\frac{1}{2}} v$$

- By inspection, $p \rightarrow m v$ as $\beta \rightarrow 0$.

- We can also do a Taylor Series expansion:

$$\frac{dp}{d\beta} = m \left(1 - \beta^2\right)^{-\frac{1}{2}} + m \beta \left(1 - \beta^2\right)^{-\frac{3}{2}} v$$

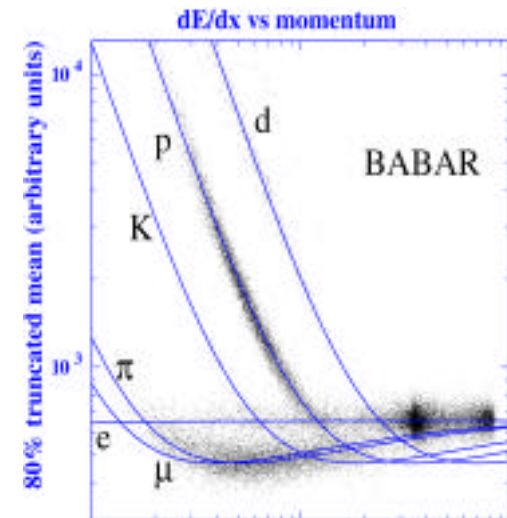
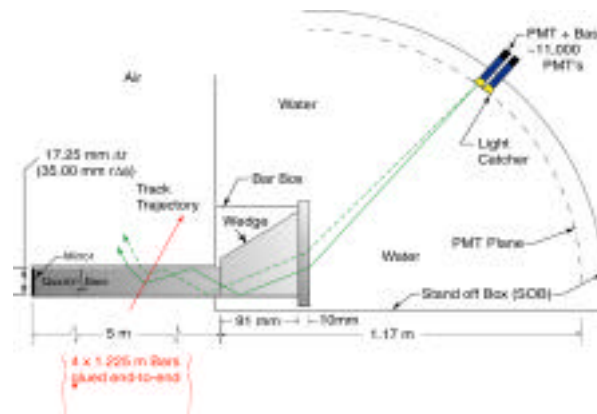
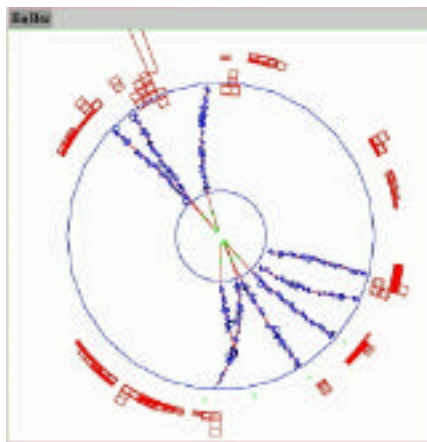
$$p = m v \left(1 + \frac{3}{2} \beta^2 + \dots\right) = m v$$

What we can measure in the Laboratory

We measure momenta of charged tracks from their radii of curvature in a magnetic field:

$$p = m \gamma v = 1 / r$$

Cerenkov light and specific ionization depend directly on the speed of a particle, β .



A Muon in the Laboratory

- Let's consider our muon moving with $\beta = 0.9999$ in the laboratory. As we calculated earlier, $\gamma = 70.71$. To calculate the energy and momentum of the muon, we need to know its (rest) mass:

$$m(\mu) = 0.106 \text{ GeV}/c^2$$

- The energy and momentum are

$$p(\mu) = 7.493 \text{ GeV}/c$$

$$E(\mu) = 7.494 \text{ GeV}$$

- For fun, we can compare this energy with the masses of particles we will encounter

$$m(e^-) = 0.511 \text{ MeV}/c^2$$

$$m(p) = 0.938 \text{ GeV}/c^2$$

$$m(\mu^-) = 0.106 \text{ GeV}/c^2$$

$$m(D^0) = 1.865 \text{ GeV}/c^2$$

$$m(\pi^+) = 0.140 \text{ GeV}/c^2$$

$$m(K^+) = 3.097 \text{ GeV}/c^2$$

$$m(K^+) = 0.494 \text{ GeV}/c^2$$

$$m(B^0) = 5.275 \text{ GeV}/c^2$$

Conservation of Energy and Momentum

- The Lorentz transformation is a linear transformation. It can be written generally as:

$$a^\mu = \Lambda^\mu{}_\nu a^\nu \quad \text{with} \quad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$$

- If conservation of energy and momentum is true in one inertial reference frame: $p_1 + p_2 = p_3 + p_4$ then

$$\Lambda^\mu{}_\nu (p_1^\nu + p_2^\nu - p_3^\nu - p_4^\nu) = p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu$$

- Conservation of momentum and energy is not required by special relativity, but it is consistent with special relativity.

Measuring Invariant Mass

- We can measure the invariant mass of a pair (collection) of particles by measuring the energy and momentum of each, and then summing to get the four-momentum of the ensemble:

$$P_{parent}^{\mu} = \sum_i P_i^{\mu}$$

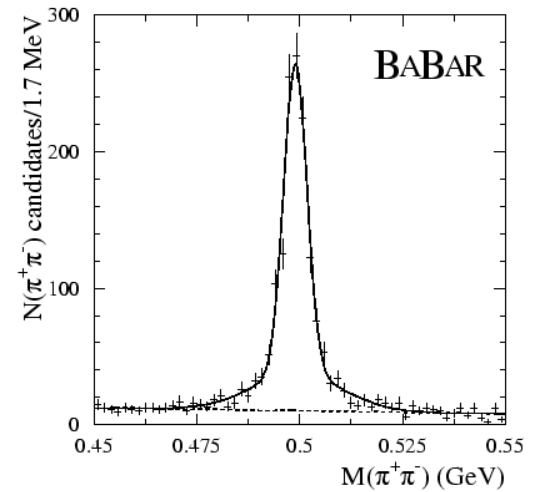
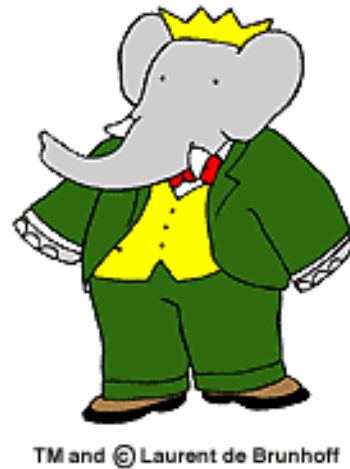
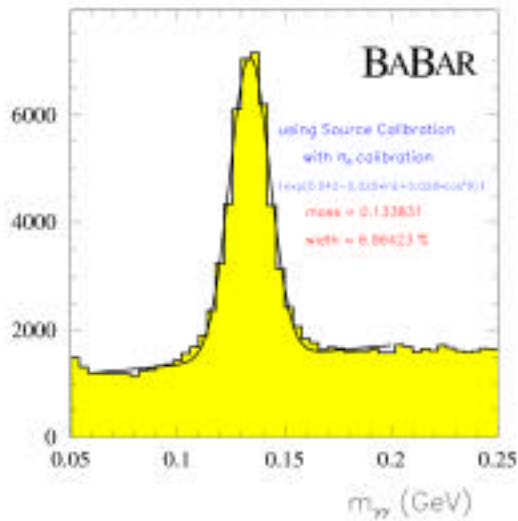
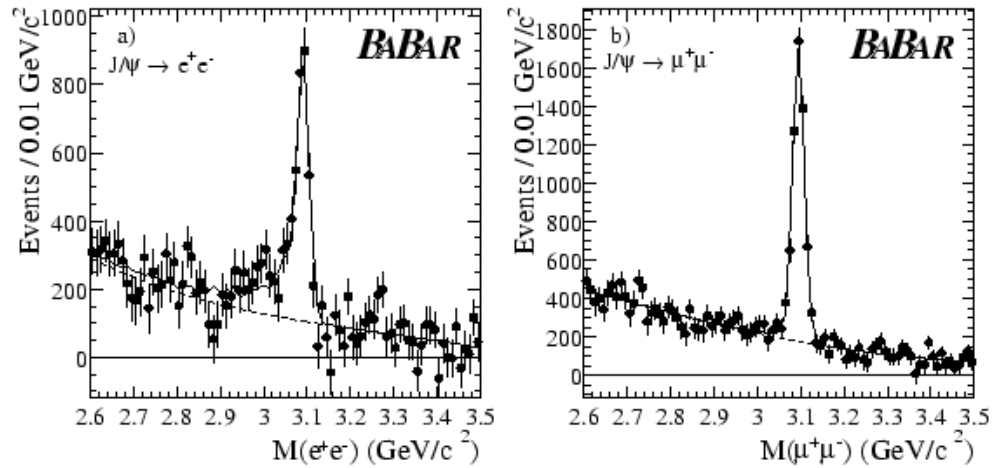
- from which we can calculate the mass of the ensemble

$$mass(ensemble) = \sqrt{P_{\mu,parent} P_{parent}^{\mu}}$$

- One rarely measures the momentum and energy of a particle directly in an experiment; rather one measures the momentum (or energy) and calculates the energy (or momentum) using

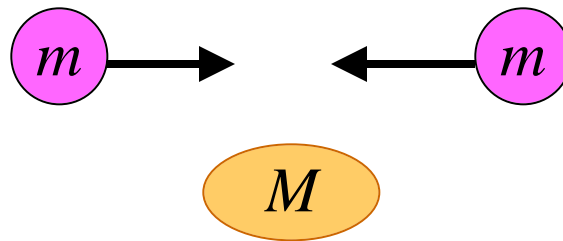
$$E^2 = p^2 + m^2$$

Invariant Mass Distributions



Collisions: Example 3.1 from Griffiths

- Two lumps of clay, each of mass m , collide head-on at $\frac{3}{5}c$. They stick together. What is the mass M of the final composite lump?



- Energy and momentum are conserved, so the invariant mass of the lump after the collision equals the invariant mass of the pair of lumps before the collision:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = 0;$$

$$E = E_1 + E_2 = 2m \gamma = \frac{2m}{\sqrt{1 - 0.6^2}} = \frac{2m}{\sqrt{.64}} = \frac{2m}{0.8} = \frac{5}{4} 2m$$

$$M^2 = E^2 - p^2 = E^2 \quad M = \frac{5}{4} 2m > 2m$$

Decay: Example 3.3 from Griffiths

First Solution

- A pion at rest decays into a muon plus a neutrino. What is the speed of the muon?

$$p = p_\mu + p$$

$$\vec{p} = \vec{p}_\mu + \vec{p} = \mathbf{0}; \quad \vec{p}_\mu = -\vec{p}$$

$$E = E_\mu + E = m$$

$$\sqrt{m_\mu^2 + p^2} + p = m \quad ; \quad m_\mu^2 + p^2 = (m^2 - 2m p + p^2)$$

$$p = \frac{m^2 - m_\mu^2}{2m}; \quad E_\mu^2 = p^2 + m_\mu^2 = \frac{m^2 + m_\mu^2}{2m}$$

$$= \frac{m}{m} = \frac{E}{p}; \quad \mu = \frac{E_\mu}{p_\mu} = \frac{m^2 - m_\mu^2}{m^2 + m_\mu^2}$$