Elastic Scattering of Electrons by Nuclei

- **We want to consider the elastic scattering of electrons by nuclei to see (i) how finite size can be accounted for, and (ii) how the electromagnetic potential (1/r) is converted to a quantum mechanical matrix element.**
- **The presentation here follows that of Perkins,** Introduction of High Energy Physics, **Third Edition, chapter 6.**
- **We will start by considering electrons and nuclei as spinless isolated particles with the nucleus at rest. We will later consider the nucleus as part of an atom, and we will review the effects of spin, etc.**

Elastic Scattering of Spinless Electrons by Nuclei - 1

- **In first order perturbation theory, the transition rate is**

$$W = \frac{2\pi}{\hbar} \, | M_{if} |^2 \, \rho_f$$

where $M_{if} = \int \varphi_f^* (\vec{r}) V(\vec{r}) \varphi_i(\vec{r}) d\vec{r}$

- **Recall,** $\psi_i = \varphi_i e^{-i\omega_i t}; \;\; \psi_f = \varphi_f e^{-i\omega_f t}$

with $\varphi_i = e^{i\vec{k}_i \vec{r}}; \;\; \varphi_f = e^{i\vec{k}_f \vec{r}}$

- **That is,** $\psi_i(\vec{r},t)$ **and** $\psi_i(\vec{r},t)$ **are plane waves.**

Elastic Scattering of Spinless Electrons by Nuclei - 2

- **Begin the calculation formally:**

$$\mathrm{M_{if}} = \int e^{-i\vec{k}_f \vec{r}} V(\vec{r}) e^{i\vec{k}_i \vec{r}} d\vec{r}$$

$$= \int e^{(\vec{k}_i - \vec{k}_f)\vec{r}} V(\vec{r}) d\vec{r}$$

$$= \int e^{\vec{q}\,\vec{r}} V(\vec{r}) d\vec{r} \qquad \textbf{where} \quad \vec{q} \equiv \vec{k}_i - \vec{k}_f$$

- **Given a charge density $\rho(\vec{R})$:**

$$\int \rho(\vec{R}) d\vec{R} = 1$$

then

$$V(\vec{r}) = \frac{Ze^2}{4\pi} \int \frac{\rho(\vec{R})}{|\vec{r} - \vec{R}|} d\vec{R}$$

Elastic Scattering of Spinless Electrons by Nuclei - 3

- **Just a bit of algebra gives:**

$$\begin{aligned} \mathrm{M}_{\mathrm{if}} &= \frac{Ze^2}{4\pi} \iint e^{i\vec{q}\,\vec{r}} \frac{\rho(\vec{R})}{|\vec{r}-\vec{R}|} d\vec{R} d\vec{r} \\ &= \frac{Ze^2}{4\pi} \int e^{i\vec{q}\,\vec{R}} \rho(\vec{R}) d\vec{R} \int \frac{e^{i\vec{q}\,(\vec{r}-\vec{R})}}{|\vec{r}-\vec{R}|} d\vec{r} \\ &= \frac{Ze^2}{4\pi} F(q^2) \int \frac{e^{i\vec{q}\,(\vec{r}-\vec{R})}}{|\vec{r}-\vec{R}|} d\vec{r} \end{aligned}$$

where we have defined the elastic scattering form factor

$$F(q^2) = \int e^{i\vec{q}\,\vec{R}} \rho(\vec{R}) d\vec{R}$$

Elastic Scattering of Spinless Electrons by Nuclei - 4

- **With $\vec{s} \equiv \vec{r} - \vec{R}$, and α the polar angle between $\vec{s}$ and $\vec{q}$**

$$M_{if} = \frac{Ze^2}{4\pi} F(q^2) \int \frac{e^{iqs\cos\alpha}\ 2\pi\ s^2 ds d\cos\alpha}{s}$$

$$= \frac{Ze^2}{2} F(q^2) \int s ds \int e^{iqs\cos\alpha}\ d\cos\alpha$$

$$= \frac{Ze^2}{2} F(q^2) \int \frac{s ds (s e^{iqs} - e^{-iqs})}{iqs}$$

$$= \frac{Ze^2}{2} F(q^2) \int \frac{2\sin qs}{q} ds$$

Elastic Scattering of Spinless Electrons by Nuclei - Add Atomic Screening

- **The term which depends on the nature of the potential:**

$$\int \frac{2\sin qs}{q} ds$$

diverges as $q^2 \to 0$ so we say that the range of the electromagnetic potential is infinite.

- **Atoms have clouds of electrons as well as nuclei, and the effect is to modify the electric potential:**

$$V'(\vec{r}) = e^{-r/a} V(\vec{r})$$

and the exponential factor is referred to as screening.

Elastic Scattering of Spinless Electrons by Neutral Atoms - 1

- **With the screening potential, the matrix element becomes:**

$$M_{if} = \frac{Ze^2}{4\pi} F(q^2) \int \frac{e^{i\vec{q}\,(\vec{r}-\vec{R})} \; e^{-r/a}}{|\vec{r} - \vec{R}|} d\vec{r}$$

The charge density $\rho(\vec{R})$ **is significantly greater than zero only for** $|\vec{R}| \sim O(10^{-13} cm)$.

- **Therefore** $e^{-|\vec{r}-\vec{R}|/a} = e^{-s/a} \rightarrow e^{-r/a}$.

recall that $a \sim O(10^{-8} cm)$.

Elastic Scattering of Spinless Electrons by Neutral Atoms - 2

$$M_{if} \to \frac{Ze^2}{4\pi} F(q^2) \int \frac{e^{iqs\cos\alpha} \; e^{-s/a} \; 2\pi \; s^2 ds\, d\cos\alpha}{s}$$

$$= \frac{Ze^2}{2} F(q^2) \int s \; e^{-s/a} \; ds \int e^{iqs\cos\alpha} \; d\cos\alpha$$

$$= \frac{Ze^2}{2} F(q^2) \int s e^{-s/a} \left[\frac{(e^{iqs} - e^{-iqs})}{iqs} \right] ds$$

$$= \frac{Ze^2}{2} \frac{F(q^2)}{iq} \int \left[e^{-s\left(\frac{1}{a} - iq\right)} - e^{-s\left(\frac{1}{a} + iq\right)} \right] ds$$

Elastic Scattering of Spinless Electrons by Neutral Atoms -2

$$\mathrm{M}_{if} = \frac{Ze^2}{2}\frac{F(q^2)}{iq}\int\left[e^{-s\left(\frac{1}{a}-iq\right)} - e^{-s\left(\frac{1}{a}+iq\right)}\right]ds$$

$$= \frac{Ze^2}{2}\frac{F(q^2)}{iq}\left[\frac{1}{\left(\frac{1}{a}-iq\right)} - \frac{1}{\left(\frac{1}{a}+iq\right)}\right]$$

$$= \frac{Ze^2}{2}\frac{F(q^2)}{iq}\left[\frac{\left(\frac{1}{a}+iq\right)-\left(\frac{1}{a}-iq\right)}{\left(\frac{1}{a}-iq\right)\left(\frac{1}{a}+iq\right)}\right]$$

Elastic Scattering of Spinless Electrons by Neutral Atoms -3

$$\mathrm{M}_{if} = \frac{Ze^2}{2}\frac{F(q^2)}{iq}\left[\frac{\left(\frac{1}{a}+iq\right)-\left(\frac{1}{a}-iq\right)}{\left(\frac{1}{a}-iq\right)\left(\frac{1}{a}+iq\right)}\right]$$

$$= \frac{Ze^2}{2}\frac{F(q^2)}{iq}\left[\frac{2iq}{\frac{1}{a^2}+q^2}\right]$$

$$= \frac{Ze^2\ F(q^2)}{q^2+1/a^2}$$

Elastic Scattering of Spinless Electrons by Neutral Atoms -4

- **For** $q^2 >> 1/a^2$,

$$M_{if} \to \frac{Ze^2 \; F(q^2)}{q^2}$$

- **When is** $q^2 >> 1/a^2$ **?**

$$k = \hbar c / a; \quad \hbar c = 197\text{MeV fm} = 197\text{MeV} \times 10^{-13}\text{cm}$$

$$a = 10^{-8}\text{cm} \qquad k = \frac{197\text{MeV} \times 10^{-13}\text{cm}}{10^{-8}\text{cm}}$$

$$\sim 2 \times 10^{-3}\text{MeV} = 2\text{keV}$$

Form Factors and Central Potentials

- **We can use the same formalism for and central potential that we used for the 1/r potential, and the form factor will emerge again as a factorizable term.**

$$V(\vec{r}) = \int \rho(\vec{R}) V(\vec{r} - \vec{R}) d\vec{R}$$

$$M_{if} = \iint e^{i\vec{q}\,\vec{r}} \rho(\vec{R}) V(\vec{r} - \vec{R}) d\vec{R} d\vec{r}$$

$$= \int e^{i\vec{q}\,\vec{r}} \rho(\vec{R}) d\vec{R} \;\; \int e^{i\vec{q}\,(\vec{r}-\vec{R})} V(\vec{r} - \vec{R}) d\vec{r}$$

$$= F(q^2) \int e^{iqs\cos\alpha} V(s) \;\; s^2 ds \;\; d\varphi \;\; d\cos\alpha$$

$$= F(q^2) 2\pi \int s V(s) \frac{2\sin(qs)}{q} \;\; ds$$

The Finite Rutherford Proton - 1

- When scattering spinless electrons from protons, we found that for $q^2 >> 1/a^2$

$$\mathrm{M}_{\mathrm{if}} \to \frac{Ze^2 \; F(q^2)}{q^2}$$

- Using the notation

  electron + nucleus $\longrightarrow$ electron + nucleus

  $\vec{p}_e$ at rest $\vec{p}_e$ $\vec{p}\,'_N$

- With the assumptions that (i) the recoiling nucleus is non-relativistic, and (ii) that the nuclear recoil momentum is small compared to the momentum of the incident electron:

$$p'_N = q << p_e$$

The Finite Rutherford Proton - 2

- **One can approximate**

$$q^2 = 2p_e^2 - 2p_e^2\cos\theta = 4p_e^2\sin^2\frac{\theta}{2}$$

so that

$$\frac{d\sigma}{d\Omega} = \frac{Z^2(e^2/4\pi)^2\left[F(q^2)\right]^2}{4p_e^2\sin^4(\theta/2)}$$

- **With** $d\Omega = 2\pi\ d(\cos\theta) = \dfrac{2\pi\ dq^2}{2p_e^2}$,

$$\frac{d\sigma}{dq^2} = \frac{4\pi\alpha^2 Z^2\left[F(q^2)\right]^2}{q^4}$$

Point-like Elastic Scattering - 1

- **Rutherford Scattering - non-relativistic quantum mechanics, first Born approximation, no spin or magnetic moments.**

- **Mott Scattering - spin 1/2 electrons, spin 0 protons, single photon exchange**

- **Dirac Scattering - spin 1/2 electrons, spin 1/2 protons, point-like → $\mu_p = e\hbar / 2m_p c$, single photon exchange.**

Point-like Elastic Scattering - 2

- **Rutherford Scattering:**

$$\left[\frac{d\sigma}{d\Omega}\right]_R = \frac{Z^2(e^2/4\pi)^2}{4p_e^2\sin^4(\theta/2)}$$

- **Mott Scattering:**

$$\left[\frac{d\sigma}{d\Omega}\right]_M = \left[\frac{d\sigma}{d\Omega}\right]_R \times \left\{\frac{\cos^2(\theta/2)}{1+\frac{2p_e}{m_p}\sin^2(\theta/2)}\right\}$$

- **Dirac Scattering:**

$$\left[\frac{d\sigma}{d\Omega}\right]_D = \left[\frac{d\sigma}{d\Omega}\right]_M \times \left\{1+\frac{q^2}{2m_p^2}\times\tan^2(\theta/2)\right\}$$

Rosenbluth Scattering - 1

- **Rosenbluth extends the Dirac formula to a finite proton:**

$$\left[\frac{d\sigma}{d\Omega}\right]_{RB} = \left[\frac{d\sigma}{d\Omega}\right]_{M} \times \left\{ \left(\frac{G_E^2 + \dfrac{q^2}{4m_p^2} G_M^2}{1 + \dfrac{q^2}{4m_p^2}} \right) + \frac{q^2}{4m_p^2} \times 2G_M^2 \times \tan^2(\theta/2) \right\}$$

where $G_E = G_E(q^2); \;\; G_M = G_M(q^2)$ **and**

$$G_E^P(0) = 1; \;\; G_E^N(0) = 0; \;\; G_M^P(0) = 2.79; \;\; G_M^N(0) = -1.91$$

Rosenbluth Scattering -2

- **To extract the form factors, one may fix q^2 and vary the scattering angle.**

$$\left(\frac{d\sigma}{d\Omega}\right) = \left\{ A(q^2) + B(q^2) \times \tan^2\theta \right\} \times \left(\frac{d\sigma}{d\Omega}\right)_M$$

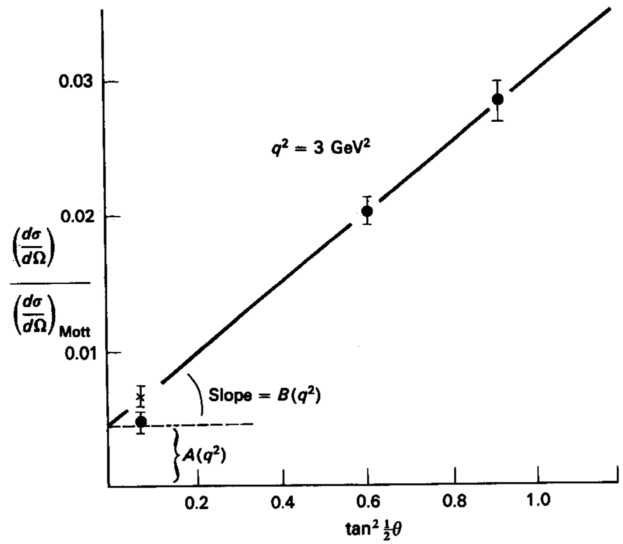


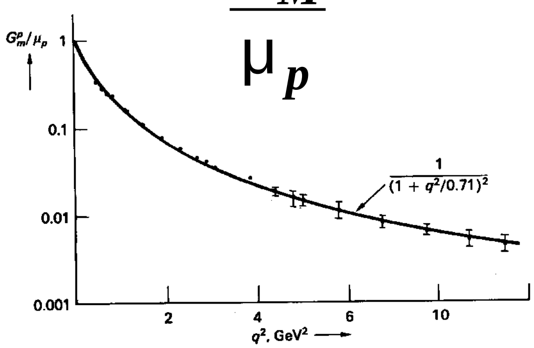
Figure 6.4 The electron-proton scattering cross-section plotted for fixed q^2 and different scattering angles θ (Rosenbluth plot). (After Weber (1967).)

Rosenbluth Scattering -3

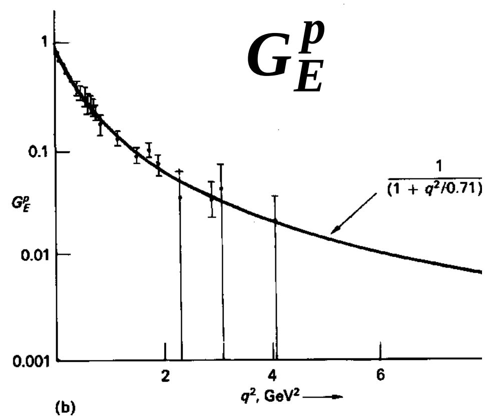
Experimentally, the form factors obey a simple scaling law:

$$G_E^p(q^2) = \frac{G_M^p(q^2)}{\mu_p} = \frac{G_M^n(q^2)}{\mu_n}; \;\; G_E^n(q^2) = 0$$

$\frac{G_M^p}{\mu_p}$



G_E^p



$\frac{G_M^n}{\mu_n}$